*Journal of Statistical Physics, Vol. 77, Nos. 1/2, 1994*

# Observation of Quantum Particles on a Large Space-Time Scale

**L. J. Landau**[1]

*Received August 9, 1993*

A quantum particle observed on a sufficiently large space-time scale can be described by means of classical particle trajectories. The joint distribution for large-scale multiple-time position and momentum measurements on a nonrelativistic quantum particle moving freely in $R^{\nu}$ is given by straight-line trajectories with probabilities determined by the initial momentum-space wavefunction. For large-scale toroidal and rectangular regions the trajectories are geodesics. In a uniform gravitational field the trajectories are parabolas. A quantum counting process on free particles is also considered and shown to converge in the large-space-time limit to a classical counting process for particles with straight-line trajectories. If the quantum particle interacts weakly with its environment, the classical particle trajectories may undergo random jumps. In the random potential model considered here, the quantum particle evolves according to a reversible unitary one-parameter group describing elastic scattering off static randomly distributed impurities (a quantum Lorentz gas). In the large-space-time weak-coupling limit a classical stochastic process is obtained with probability one and describes a classical particle moving with constant speed in straight lines between random jumps in direction. The process depends only on the ensemble value of the covariance of the random field and not on the sample field. The probability density in phase space associated with the classical stochastic process satisfies the linear Boltzmann equation for the classical Lorentz gas, which, in the limit $h \to 0$, goes over to the linear Landau equation. Our study of the quantum Lorentz gas is based on a perturbative expansion and, as in other studies of this system, the series can be controlled only for small values of the rescaled time and for Gaussian random fields. The discussion of classical particle trajectories for nonrelativistic particles on a macroscopic space-time scale applies also to relativistic particles. The problem of the spatial localization of a relativistic particle is avoided by observing the particle on a sufficiently large space-time scale.

**KEY WORDS:** Lorentz gas; Boltzmann equation; Landau equation; Van Hove limit; weak-coupling limit; random potential; quantum trajectories.

[1] Department of Mathematics, King's College London, London WC2R 2LS, United Kingdom.



0022-4715/94/1000-0259$07.00/0 

## 1. INTRODUCTION

A limiting procedure applied to the mathematical model of a physical system is useful in isolating the dominant behavior of the system in a given regime of parameters describing the state of the system, its interactions, and the method of observation. The infinite-volume limit is the appropriate vehicle in studying the phase transitions of a large system in equilibrium. Similarly, use of a large space and/or time scale combined with an appropriate scaling of interactions can lead to hydrodynamic behavior of a particle system, irreversible behavior of a reversible system, or classical behavior of a quantum system. Limiting procedures can thus bridge the gap between a general fundamental theory and the various special theories which apply only in certain circumstances, and can lead to a deeper understanding of the fundamental theory in terms of the often intuitive and better understood special theories.

Decoherence of quantum probability amplitudes can occur over large space-time distances. On a large enough scale, the joint distribution of multiple-time position and momentum measurements on a free quantum particle is given by a classical probability distribution on straight-line trajectories. On a large-scale manifold the free evolution should be described by a mixture of geodesic trajectories. Computations are presented for tori and flat rectangular regions. The experimental arrangement will determine what constitutes a sufficiently large scale. For example, a laboratory interferometer exhibits quantum coherence for appropriate initial particle states and would have to be rescaled for decoherence to set in. Put another way, the large-space-time limit considered here rests on strong operator convergence on the Hilbert space of wavefunctions in the case of the free evolution, and weak operator convergence in the case of evolution in a random potential. In contrast to norm convergence, this convergence is *nonuniform*. This nonuniformity is perhaps best emphasized, as in ref. 12, by a quotation from Bell,[1] which we paraphrase as follows:

> While for any given wavefunction one can find a space-time scaling for which the deviation from classical behavior is as small as one likes, for any given space-time scaling one can find a wavefunction for which it is as big as one does *not* like.

A quantum particle observed on a large scale and interacting sufficiently weakly with its environment should be described by a mixture of classical straight-line trajectories subjected to random influences, as may be seen in certain circumstances in bubble chamber tracks. The model considered here consists of a quantum particle scattering off static randomly distributed impurities represented by a random potential. The particle evolves according to the reversible unitary one-parameter group $e^{-i(t/\hbar)H}$ generated by the Hamiltonian $H = H_0 + \lambda V$, where $H_0 = \mathscr{P}^2/(2m)$, $V = v(q)$, and $\lambda$ is a coupling constant. Here $\mathscr{P}$ is the momentum operator, $q$ is the position operator, and $v(x)$ is the potential generated by the impurities. In the ensemble describing the random distribution of impurities, $v(x)$ is a random field which will be supposed translation invariant: $\langle v(x_1+a) \cdots v(x_n+a) \rangle = \langle v(x_1) \cdots v(x_n) \rangle$.

The behavior of a single quantum particle in a random potential in the large-time weak-interaction limit (Van Hove limit) has been studied by Martin and Emch,[17] Spohn,[19] Dell'Antonio,[6] and Ho *et al.*[12] In this limit the coupling constant $\lambda \to 0$ and the rescaled time $\tau = \lambda^2 t$ is introduced. The spatial coordinates are not rescaled. Thus the system is considered on a microscopic space scale and a macroscopic time scale. Since the position of the particle moves off to infinity as $t \to \infty$, only the momentum observable is studied. In contrast, in the present study the space coordinates will also be rescaled so that the system is considered on a macroscopic space and time scale. It will be shown that the joint distribution of the outcomes of successive position and momentum measurements on the particle, containing the quantum disturbance of one measurement on another as expressed by the noncommutativity of observables, converges in the large-space-time, weak-coupling limit to the joint distribution of a classical stochastic process describing a classical point particle moving with constant speed in straight lines between random jumps in direction.

The large-time, weak-interaction limit for an infinitely extended Fermi gas in a random potential is studied by Ho *et al.*[12, 15] The local microscopic quantum state of the gas is considered on a macroscopic time scale and the irreversible semigroup describing the evolution of the quantum state and yielding increase of entropy density is obtained. The system behaves in this limit like an open system due to particles moving to and from infinity as $t \to \infty$. The situation is different in the present study, where the gas is considered on a macroscopic spatial scale. We consider a number $N$ of particles and can consider the limit $N \to \infty$. The system has a finite macroscopic density and corresponds to a low-density limit of the system studied in ref. 12.

The perturbative method used here to study the random potential model can be controlled, as in the other studies of the model, only for small rescaled time $\tau$ and Gaussian random fields.

**Particle Tracks in Quantum Theory**. The appearance of particle tracks, as in Wilson cloud chamber photographs, has been studied and understood from the inception of the quantum theory. Heisenberg gives a discussion in his book, *The Physical Principles of the Quantum Theory*, which appeared in 1930.[2] Discussing the Wilson photographs, he states (ref. 9 §V.1), "It is always correct to apply the classical theory to such semi-macroscopic phenomena, and the quantum theory is necessary only for the explanation of the finer features." The motion of wave packets in uniform gravitational and magnetic fields was considered by Darwin in 1927.[(4)]

In these and many related studies the *approximate* motion of wave packets is discussed. In contrast to this, in the present study it is shown that the large-space-time limit provides a convenient mathematical technique for the derivation of *precise* particle trajectories for free quantum particles, for quantum particles on manifolds, and for quantum particles weakly interacting with a random environment.

## 1.1. Rescaling Space, Time, and Interaction Strength

A classical particle moving freely in $R^\nu$ moves on a straight-line trajectory $q(t) = q + (t/m)\mathscr{P}$, $q$ and $\mathscr{P}$ being the initial position and momentum, respectively. With respect to the rescaled time $\tau = \lambda^2 t$, $q(\tau) = q + \lambda^{-2}(\tau/m)\mathscr{P}$ and in the limit $\lambda \to 0$ the particle moves off to infinity. Introducing a rescaled position $\mathscr{Q} = \lambda^2 q$ leads to $\mathscr{Q}(\tau) = \lambda^2 q + (\tau/m)\mathscr{P}$ and in the limit $\lambda \to 0$, $\mathscr{Q}(\tau) = (\tau/m)\mathscr{P}$. Furthermore, shifting the initial position of the particle by $x_0 = X_0/\lambda^2$ gives in the limit $\mathscr{Q}(\tau) = X_0 + (\tau/m)\mathscr{P}$. The equations for a quantum particle are the same: In the Heisenberg picture the position operator of the particle at time $t$ is $q(t) = q + (t/m)\mathscr{P}$, where $q$ and $\mathscr{P}$ are the time-zero position and momentum operators, which do not commute. Defining the rescaled time $\tau$ and rescaled position $\mathscr{Q}$ as above and shifting the initial quantum state by $X_0/\lambda^2$ leads to $\mathscr{Q}(\tau) = \lambda^2 q + X_0 + (\tau/m)\mathscr{P}$, which converges to $\mathscr{Q}(\tau) = X_0 + (\tau/m)\mathscr{P}$. Notice that in the large-space-time limit the quantum position operator $q(t)$ goes over to $X_0 + (\tau/m)\mathscr{P}$, which is a function only of the momentum operator $\mathscr{P}$, and hence they mutually commute at different $\tau$ values. The joint distribution for large-scale position and momentum measurements may then be represented by a classical probability distribution for the initial position $X_0$ and the observable $\mathscr{P}$, the particle moving on the classical trajectories $X_0 + (\tau/m)\mathscr{P}$.

Now consider a classical particle moving through an environment of randomly distributed scatterers. The speed $v$, mean free path $l$, and mean free time $T$ are related by $l = vT$. Let $\rho$ be the density of scatterers. If $\sigma$ is the total cross section of a scatterer, then $\sigma l \rho = 1$, so $l = 1/\rho\sigma$, $T = 1/\rho v\sigma$. In time $t$ there are $N = t/T$ collisions. The distance traveled between collisions is $l$ and supposing random directions of travel, the average square of the total displacement is $Nl^2 = (t/T)l^2 = vt/\rho\sigma = Dt$, where the diffusion con-

[2] Heisenberg's treatment is the same as Mott's.[(16)]

stant $D = v/\rho\sigma$. Now suppose the interaction strength is measured by $\lambda$. Then the cross section will go like $\sigma = \lambda^2\omega$ and hence $D = \lambda^{-2}v/\rho\omega$. The displacement is then $(Dt)^{1/2} = (v/\rho\omega)^{1/2}\lambda^{-1}\sqrt{t} = \lambda^{-2}(v/\rho\omega)^{1/2}\sqrt{\tau}$ in terms of the rescaled time $\tau = \lambda^2 t$. Rescaling spatial coordinates as above yields a large-scale displacement of $(v/\rho\omega)^{1/2}\sqrt{\tau}$. Thus the same large scaling of space-time is appropriate for a free particle and a particle interacting weakly with its environment. (This is important in a perturbative treatment since the zeroth-order term is the free evolution.)

The scaling of space, time, and interaction strength plays an important role in studies of the macroscopic properties of many physical systems.[20]

## 1.2. Multiple-Time Position Measurements on a Quantum Particle

A measurement of the position of the particle at a particular time will disturb the particle and affect subsequent measurements. Consequently the joint distribution for the observed position of the particle at various times will depend on the nature of the apparatus used in the measurements. (The particle may even be absorbed by the apparatus, making subsequent measurements on the particle impossible.) Ideally the detailed experimental arrangement should be modelled[10, 26] within the quantum theory, but the framework of quantum measurement theory provides a convenient mathematical formulation.[5, 13] The result of a measurement is described by an "operation" on the states of the system. A standard prescription giving the operation in the case of "ideal" measurements is the "wave packet reduction formula" or "projection postulate."[13] In the case of position measurements it takes the following form. Let $q(t)$ be the self-adjoint operator corresponding to the position of the particle at time $t$, and let $\psi$ denote the wave function of the particle. Given a region $\Delta \subset R^\nu$, the probability that the particle is found in $\Delta$ at time $t$ is $(\psi, \chi_\Delta(q(t))\psi)$, where $\chi_\Delta(x) = 1$ if $x \in \Delta$ and $=0$ otherwise. [The operator $\chi_\Delta(q(t))$ projects onto the subspace of wavefunctions for which the particle is with certainty in $\Delta$ at time $t$.] Given that the particle is found in $\Delta$ at time $t$, the subsequent state of the particle is taken to be $\|\chi_\Delta(q(t))\psi\|^{-1}\chi_\Delta(q(t))\psi$.[3] Consequently the probability that the particle is found in $\Delta'$ at time $t'$ given that it was found in $\Delta$ at time $t$ is

$$(\chi_\Delta(q(t))\psi, \chi_{\Delta'}(q(t'))\chi_\Delta(q(t))\psi)\ \|\chi_\Delta(q(t))\psi\|^{-2}$$

[3] We are considering an ideal coarse-grained measurement with coarse graining described by cells such as $\Delta$, and not an ideal finer-grained measurement, which would lead to the same probability that the particle is in $\Delta$, but for which a subsequent application of the projection onto $\Delta$ would be inappropriate.

Thus the joint probability that the particle is in $\Delta$ at time $t$ and in $\Delta'$ at time $t'$ is

$$(\psi, \chi_{\Delta}(q(t))\, \chi_{\Delta'}(q(t'))\, \chi_{\Delta}(q(t))\psi)$$

and the state of the particle after the second position measurement is

$$\|\chi_{\Delta'}(q(t'))\, \chi_{\Delta}(q(t))\psi\|^{-1}\, \chi_{\Delta'}(q(t'))\, \chi_{\Delta}(q(t))\psi.$$

In this way we arrive at the following formula for the joint probability that the particle is found in the region $\Delta_1$ at time $t_1$, and then in $\Delta_2$ at time $t_2$,..., and then $\Delta_N$ at time $t_N$, where $t_1 < t_2 < \cdots < t_N$:

$$\begin{aligned} &P_{\psi}(\Delta_1, t_1;...; \Delta_N, t_N) \\ &\quad = (\psi, \chi_{\Delta_1}(q(t_1))\, \chi_{\Delta_2}(q(t_2)) \cdots \chi_{\Delta_N}(q(t_N)) \cdots \chi_{\Delta_2}(q(t_2))\, \chi_{\Delta_1}(q(t_1))\psi) \end{aligned} \tag{1}$$

Notice that

$$\begin{aligned} &P_{\psi}(\Delta_1, t_1; \Delta_2, t_2;...; \Delta_N, t_N) + P_{\psi}(\Delta_1^c, t_1; \Delta_2, t_2;...; \Delta_N, t_N) \\ &\qquad \neq P_{\psi}(\Delta_2, t_2; ...; \Delta_N, t_N) \end{aligned}$$

where $\Delta^c$ is the complement in $R^{\nu}$ of $\Delta$. This expresses the disturbance of the position measurement at time $t_1$ on the subsequent position measurements, and is a typical consequence of the noncommutativity (complementarity) of observables in quantum theory. Consequently the *usual* expression for joint position probabilities given by a classical stochastic process will not reproduce the quantum joint distribution. Nevertheless we shall show that in the large-space-time limit the *quantum* joint distribution will converge to those of a *classical* stochastic process.

Actually a more general formulation within quantum measurement theory than the above projection postulate for position measurements will be necessary for the application to the weak-coupling limit in a random potential, since the estimates require a certain smoothness of the functions and $\chi_{\Delta}$ is not smooth. We shall give a partition $\{\Delta_j\}$ of $R^{\nu}$ and assign to each $\Delta_j$ a smooth nonnegative function $\eta_{\Delta_j}$ which vanishes outside a small neighborhood of $\Delta_j$ and $=1$ in the interior of $\Delta_j$ except in a small neighborhood of the boundary of $\Delta_j$. Thus $\eta_{\Delta_j}$ is a smooth approximation to $\chi_{\Delta_j}$. The probability that the particle is found in $\Delta_{j_1}$ at time $t_1$,..., and in $\Delta_{j_N}$ at time $t_N$ is given by a similar formula[4] to Eq. (1):

[4] Formula (2) may be thought of as modeling position measurements by an apparatus with a *smooth* response to the presence of a quantum particle, rather than the *sharp* response associated with Eq. (1). In particular, according to Eq. (1), given a position measurement at time $t_1$ registering the particle in the cell $\Delta$, an immediate subsequent measurement ($t_2 \to t_1$) will with probability 1 register the particle in the *same* cell, whereas according to Eq. (2) the subsequent measurement may register the particle in a *neighboring* cell.

$$P_\psi(\Delta_{j_1}, t_1; \ldots; \Delta_{j_N}, t_N) = (\psi, \eta_{\Delta_{j_1}}(q(t_1))\, \eta_{\Delta_{j_2}}(q(t_2)) \cdots \eta_{\Delta_{j_N}}(q(t_N)) \times \eta_{\Delta_{j_N}}(q(t_N)) \cdots \eta_{\Delta_{j_2}}(q(t_2))\, \eta_{\Delta_{j_1}}(q(t_1))\psi) \quad (2)$$

In order that well-defined probabilities are obtained in this way it is necessary that $\sum_j \eta^2_{\Delta_j}(x) = 1$ for all $x$. It is easy to construct such partitions and functions in higher dimensions by taking products of one-dimensional quantities. In one dimension consider a partition of $R^1$ into intervals which are integer translates of $\Delta = [-1/2, 1/2]$: $\Delta_j = [j - 1/2, j + 1/2]$. Let $\beta(x)$ be an infinitely differentiable function $>0$ if $|x| < 1/2 + \varepsilon$ and 0 otherwise. For example,

$$\beta(x) = \exp[-(x + 1/2 + \varepsilon)^{-2} - (x - 1/2 - \varepsilon)^{-2}] \qquad \text{if} \qquad |x| < 1/2 + \varepsilon$$
$$= 0 \qquad \text{otherwise}$$

The function $\beta_{\Delta_j}$ is a translate of $\beta$. Define $\eta_{\Delta_j}(x) = \beta_{\Delta_j}(x)/[\sum_j \beta_{\Delta_j}(x)^2]^{1/2}$. Then $\eta_{\Delta_j}(x) = 1$ for $j - 1/2 + \varepsilon < x < j + 1/2 - \varepsilon$ and $=0$ for $x \leqslant j - 1/2 - \varepsilon$ and $x \geqslant j + 1/2 + \varepsilon$. Arbitrarily fine partitions may be obtained by replacing $\beta(x)$ by $\beta(bx)$, where $b$ is large. Note that we may interpret $\eta^2_{\Delta_j}(x)$ as the probability, given that the particle is "actually at the point $x$," that its position will be measured to be in $\Delta_j$, which is consistent with $\sum_j \eta^2_{\Delta_j}(x) = 1$. This interpretation is useful when comparing with te classical joint distributions.

**Remarks.** 1. Our discussion is formulated in terms of position measurements, although the same methods apply to joint multiple-time position and momentum measurements.

2. In the case of the rescaled *free* evolution, $F(\mathscr{Q}(\tau))$ converges *strongly* to $F(X_0 + (\tau/m)\mathscr{P})$ and hence the large-scale limit can be taken for functions of position *in any time order*.

3. Although our discussion of a quantum particle in a random potential is formulated in terms of the functions $\eta_\Delta(\mathscr{Q}(\tau))$ and expressions such as (2), our methods apply to any functions $F(\mathscr{Q}(\tau))$, where $F$ is the Fourier transform of a bounded complex measure, in particular $F = 1$, and to expressions of the form

$$(\psi, F_1(\mathscr{Q}(\tau_1)) \cdots F_N(\mathscr{Q}(\tau_N))\, F'_N(\mathscr{Q}(\tau_N)) \cdots F'_1(\mathscr{Q}(\tau_1))\psi)$$

with $0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_N$. Hence we can deal with totally time-ordered expressions such as

$$F_1(\mathscr{Q}(\tau_1)) \cdots F_N(\mathscr{Q}(\tau_N))$$

or

$$F_N(\mathscr{Q}(\tau_N))\cdots F_1(\mathscr{Q}(\tau_1))$$

Such totally time-ordered expressions are considered by Dell'Antonio[6] for functions of the momentum, where, however, he does not treat expressions of the required form (2).[5]

### 1.3. Strong and Weak Convergence

We comment here on the difference between strong operator convergence for the free evolution, and weak operator convergence for the random potential evolution. In the case of the random potential, $F(\mathscr{Q}(\tau))$ converges weakly to $(T_\tau F)(X_0, \mathscr{P})$, where for any function $f(x, p)$ on phase space, $(T_\tau f)(x, p)$ is the conditional expectation of $f(x(\tau), p(\tau))$ given the time-zero values $(x(0)=x,\ p(0)=p)$ with respect to a classical stochastic process $(x(\tau), p(\tau))$ on phase space. In the case of the free evolution

$$(T_\tau^0 f)(x, p) = f(x + (\tau/m)\,p, p)$$

$T_\tau^0$ is a deterministic flow on phase space. Thus

$$T_\tau^0(FG) = (T_\tau^0 F)(T_\tau^0 G)$$

This multiplicative property is *not* satisfied by the conditional expectation $T_\tau$ of an indeterministic process.

Now if $A_\tau \to A$ weakly and $A_\tau^* A_\tau \to A^* A$ weakly, it follows that $A_\tau \to A$ *strongly*. Indeed,

$$\|(A_\tau - A)\psi\|^2 = (\psi, A_\tau^* A_\tau \psi) + (\psi, A^* A\psi) - (A\psi, A_\tau \psi) - \overline{(A\psi, A_\tau \psi)} \to 0$$

*The multiplicative (deterministic) property of $T_\tau^0$ converts weak convergence to strong convergence.*

### 1.4. Further Remarks on the Literature

1. Spohn[19] and Dell'Antonio[6] study the momentum observable for a quantum particle in a random potential. Only the average over the random potential is considered, whereas in the present study and ref. 12, results are obtained for sample random potentials, with probability one.

2. Spohn[19] studies the momentum observable only at one time and thus a momentum process is not obtained, whereas in the present study

[5] See ref. 12 for a discussion of Dell'Antonio's analysis.

multiple-time correlations are studied. Dell'Antonio[6] considers totally time-ordered multiple-time correlations, but as mentioned in the preceding remark, these do not correspond to multiple-time measurements. In addition, Dell'Antonio's analysis rests on several erroneous estimates. (See ref. 12, §7.)

3. The different character of the classical and quantum motion in a random potential arises from the effect of the interaction strength $\lambda$ on an individual scattering process. Classically, a weak potential will produce only a small deviation in the velocity. Quantum mechanically, the deviation is of order unity but with probability proportional to $\lambda^2$, as follows from the Born approximation. This difference corresponds to the transition from the linear Boltzmann equation to the linear Landau equation as $\hbar \to 0$ (Section 3.6). This point is discussed by Balescu (ref. 2, p. 599).

4. The equations of motion for a particle in a potential $V$ are

$$\frac{dq(t)}{dt} = p(t) \qquad \text{and} \qquad \frac{dp(t)}{dt} = F(q(t))$$

where $F(x) = -\nabla V(x)$. Scaling the interaction strength by $\lambda$ replaces $V$ by $\lambda V$ and scaling space and time by $\lambda^2$ gives $\tau = \lambda^2 t$ and $Q(\tau) = \lambda^2 q(\lambda^{-2}\tau)$. Set $P(\tau) = p(\lambda^{-2}\tau)$. The equations of motion become

$$\frac{dQ(\tau)}{d\tau} = P(\tau) \qquad \text{and} \qquad \frac{dP(\tau)}{d\tau} = \lambda^{-1}F(\lambda^{-2}Q(\tau))$$

These equations are equivalent to leaving space-time unscaled but scaling the potential to $\lambda V(\lambda^{-2}x)$. However, it is also necessary to consider the initial conditions $Q(0)$, $P(0)$, and in particular the communication relations $[Q(0), P(0)] = i\lambda^2\hbar$. The effective Planck constant $\lambda^2\hbar$ tends to zero as $\lambda \to 0$ and hence this is a type of classical limit $\hbar \to 0$. The classical limit of quantum theory as an $\hbar \to 0$ limit has been studied in various forms in the literature. This limit is not precisely defined, as the behavior of operators, wavefunctions, and parameters needs to be specified and depends on the physical basis for the observed classical behavior. A different version of the classical limit from the one considered here is given by Hepp.[11] The weak-coupling limit for a classical particle in a random potential (classical Lorentz gas) is studied by Kesten and Papanicolau[14] and Dürr *et al.*[7]

**Notation.** Constants will be absorbed into the definition of the Fourier transform where convenient to yield simple expressions.

## 2. FREE EVOLUTION ON A LARGE-SPACE-TIME SCALE

In this section we shall use Eq. (1) for the joint distribution of multiple-time position measurements. We shall consider the free evolution of a quantum particle in $R^{\nu}$, in rectangular boxes, and on tori. The initial state of the particle is given by the wavefunction $\psi$, translated by the rescaled amount $X_0 = \lambda^2 x_0$. We shall take the initial state to be $\psi$ and translate the observables by $x_0 = \lambda^{-2} X_0$. Then the initial position observable is $q + x_0$ and at time $t$ the position is $q + x_0 + (t/m)\mathscr{P}$. The rescaled position observable is then $\mathscr{Q}(\tau) = \lambda^{-2} q + X_0 + (\tau/m)\mathscr{P}$, which converges strongly[6] to $X_0 + (\tau/m)\mathscr{P}$ as $\lambda \to 0$. More precisely, $\mathscr{Q}(\tau)\psi$ converges in the Hilbert space norm to $[X_0 + (\tau/m)\mathscr{P}]\psi$ on a dense set of analytic vectors $\psi$ for $X_0 + (\tau/m)\mathscr{P}$, for example, on shifted Gaussian wavefunctions, and hence, since $[X_0 + (\tau/m)\mathscr{P}]$ is essentially self-adjoint on the finite linear span of such wavefunctions, for every bounded measurable function which is continuous except on a closed set of Lebesgue measure zero, $F(\mathscr{Q}(\tau))$ will converge strongly to $F(X_0 + (\tau/m)\mathscr{P})$.[8] Furthermore, if $F_1,\ldots, F_N$ are such functions, then $F_1(\mathscr{Q}(\tau_1)) \cdots F_N(\mathscr{Q}(\tau_N))$ will converge strongly to $F_1(X_0 + (\tau_1/m)\mathscr{P}) \cdots F_N(X_0 + (\tau_N/m)\mathscr{P})$. Consequently the joint probability [Eq. (1)] will converge as $\lambda \to 0$ to

$$
\begin{aligned}
&(\psi, \chi_{\Delta_1}(X_0 + (\tau_1/m)\mathscr{P}) \cdots \chi_{\Delta_N}(X_0 + (\tau_N/m)\mathscr{P}) \cdots \chi_{\Delta_1}(X_0 + (\tau_1/m)\mathscr{P})\psi) \\
&\quad = (\psi, \chi_{\Delta_1}(X_0 + (\tau_1/m)\mathscr{P}) \cdots \chi_{\Delta_N}(X_0 + (\tau_N/m)\mathscr{P})\psi) \\
&\quad = \int dp\, |\hat{\psi}(p)|^2 \chi_{\Delta_1}(X_0 + (\tau_1/m)p) \cdots \chi_{\Delta_N}(X_0 + (\tau_N/m)p) \qquad (3)
\end{aligned}
$$

Equation (3) is just the classical probability that a particle will be found in $\Delta_1$ at time $\tau_1,\ldots,$ and in $\Delta_N$ at time $\tau_N$, given that the particle follows the trajectory $X_0 + (\tau/m)p$ with probability density $|\hat{\psi}(p)|^2$. Thus the quantum joint distribution of multiple-time position measurements in the limit of large-space-time scaling may be interpreted by a classical probability distribution on classical particle trajectories.

**Remark.** Consider a wavefunction which is a coherent superposition of two wavefunctions translated by *different* amounts $X_1$ and $X_2$: $a_1 U(\lambda^{-2}X_1)\psi_1 + a_2 U(\lambda^{-2}X_2)\psi_2$. Computing the joint distribution for multiple-time position measurements as above, we obtain a sum of terms from each of the wavefunctions and cross-terms of the form $(\psi_1, U(\lambda^{-2}(X_2 - X_1)) A\psi_2)$, where $A$ is an operator depending on the rescaled position $\mathscr{Q}(\tau)$ and $A\psi_2$ converges as $\lambda \to 0$. The cross-term is then

[6] $A_\lambda$ converges strongly to $B$ if $\|A_\lambda\psi - B\psi\| \to 0$ for all wavefunctions $\psi$.

seen to tend to zero since $U(x)$ converges weakly[7] to zero as $x \to \infty$. To show this, suppose two wavefunctions $\phi_1$ and $\phi_2$ are zero outside some bounded set. Then $\phi_1$ and $U(x)\phi_2$ are orthogonal for $x$ sufficiently large. Since any wavefunction may be approximated by such $\phi$, it follows that $\lim_{x\to\infty}(\phi_1, U(x)\phi_2)=0$ for all $\phi_1, \phi_2$. Hence a *mixture* of $\psi_1$ at $X_1$ and $\psi_2$ at $X_2$ results.

## 2.1 Free Evolution on a Half-Line

The large-scale limit for a free quantum particle restricted to the half-line $[0, \infty)$ again leads to classical particle trajectories, which in this case reflect back from the origin. By using the method of images, the free evolution on $(-\infty, \infty)$ previously studied may be used in the computations.

**Method of Images.** Denote by $\mathscr{H}_D$ the Hilbert space $L^2([0,\infty), dx)$ of wavefunctions of the particle and let $\mathscr{H} = L^2((-\infty, \infty), dx)$. The free evolution on $\mathscr{H}$ is generated by the Hamiltonian $H = \mathscr{P}^2/(2m)$. Any wavefunction in the domain of $H$ is continuous and has a continuous first derivative. On smooth functions, $H = -h^2/(2m)\, d^2/dx^2$. The sub-Hilbert space $\mathscr{H}_0 \subset \mathscr{H}$ of *odd* functions is invariant under the free evolution $\exp[-i(t/h)H]$. Let $A$ be the unitary map of $\mathscr{H}_D$ onto $\mathscr{H}_0$ given by $(A\psi)(x) = \text{sign}(x)\, 2^{-1/2}\psi(|x|)$. The inverse unitary map $B$ of $\mathscr{H}_0$ onto $\mathscr{H}_D$ is $(B\psi)(x) = 2^{1/2}\psi(x)$. A continuous one-parameter group of unitary operators on $\mathscr{H}_D$ may be defined by $B\exp[itH]\,A = \exp[itH_D]$, which defines the self-adjoint operator $H_D$ on $\mathscr{H}_D$. The operator $H_D$ is just $-h^2/(2m)\, d^2/dx^2$ with the *Dirichlet* boundary condition $f(0)=0$. [If the *even* subspace of $\mathscr{H}$ had been used, then the *Neumann* boundary condition $df/dx(0)=0$ would have resulted, with the same conclusions applying.] The position operator $q_D$ on $\mathscr{H}_D$ is multiplication by $x$ and $q_D = B\,|q|\,A$. Thus

$$\begin{aligned} q_D(t) &= \exp[itH_D]\, q_D \exp[-itH_D] \\ &= B\exp[itH]\,|q|\,\exp[-itH]\,A = B|q(t)|\,A \end{aligned}$$

The rescaled position operator is $\mathscr{Q}_D(\tau) = \lambda^2 q_D(\lambda^{-2}\tau)$ and

$$\chi_{\Delta_1}(\mathscr{Q}_D(\tau_1)) \cdots \chi_{\Delta_N}(\mathscr{Q}_D(\tau_N)) \cdots \chi_{\Delta_1}(\mathscr{Q}_D(\tau_1)) = B\mathscr{F}A$$

where

$$\mathscr{F} = \chi_{\Delta_1}(|\mathscr{Q}(\tau_1)|) \cdots \chi_{\Delta_N}(|\mathscr{Q}(\tau_N)|) \cdots \chi_{\Delta_1}(|\mathscr{Q}(\tau_1)|)$$

[7] $A_\lambda$ converges weakly to $B$ if $(\psi_1, A_\lambda\psi_2) \to (\psi_1, B\psi_2)$ for all wavefunctions $\psi_1, \psi_2$.

We thus obtain for the joint distribution rescaled multiple-time position measurements

$$P_\psi(\Delta_1, \tau_1; \dots ; \Delta_N, \tau_N) = (A\psi, \mathscr{F} A\psi)$$

Now write $A\psi = 2^{-1/2}(\psi' - \psi'')$, where $\psi'(x) = \psi(x)$ if $x \in R^+$ and zero otherwise, and $\psi''(x) = \psi(-x)$ if $x \in R^-$ and zero otherwise. Since the operator $\mathscr{F}$ is even under $x \to -x$, the above expression may be rewritten as

$$(\psi', \mathscr{F}\psi') = (\psi'', \mathscr{F}\psi') \tag{4}$$

Suppose now that the particle is translated to some large-scale point $X_0 > 0$. Then $\psi'$ is replaced by $U(\lambda^{-2}X_0)\psi'$ and $\psi''$ is replaced by $U(-\lambda^{-2}X_0)\psi''$. We may now use the preceding remark to conclude that the cross-term in Eq. (4) tends to zero in the large-scale limit $\lambda \to 0$.[8] We finally obtain for the limiting joint distribution of rescaled multiple-time position measurements

$$(\psi', \mathscr{F}\psi') \tag{5}$$

where in the expression for $\mathscr{F}$, $|\mathscr{Q}(\tau)|$ is replaced by $|X_0 + (\tau/m)\mathscr{P}|$. *The presence of the absolute value has the effect of reflecting the straight-line trajectories at the origin.*

## 2.2. Free Evolution on a Circle

We shall identify the Hilbert space $\mathscr{H}_\lambda$ of wavefunctions of a particle on a circle of circumference $L/\lambda^2$ with $L^2([-\lambda^{-2}L/2, \lambda^{-2}L/2], dx)$ and thus consider it as a sub-Hilbert space of $\mathscr{H} = L^2((-\infty, \infty), dx)$. As large-scale space coordinate (angular variable) on the circle we shall take $\exp[i2\pi\lambda^2 x/L]$, which traverses the unit circle in the complex plane as $x$ varies from $-\lambda^{-2}L/2$ to $\lambda^{-2}L/2$. The large-scale quantum mechanical position operator on $\mathscr{H}_\lambda$ is then the unitary operator $\mathscr{Q}_\lambda = \exp[i2\pi\lambda^2 q_\lambda/L]$, where $q_\lambda$ is multiplication by $x$ on $\mathscr{H}_\lambda$. The momentum operator $\mathscr{P}_\lambda$ generates translations around the circle: $\mathscr{P}_\lambda = (h/i)\, d/dx$ with periodic boundary conditions at $-\lambda^{-2}L/2$ and $\lambda^{-2}L/2$. Then

$$\exp[i(a/h)\mathscr{P}_\lambda]\, \mathscr{Q}_\lambda \exp[-i(a/h)\mathscr{P}_\lambda] = \exp[i2\pi a\lambda^2/L]\, \mathscr{Q}_\lambda$$

[8] If the particle is not translated by a nonzero macroscopic amount, so that it is "at $X = 0$," then $\psi'$ is to be replaced by $A\psi$ in Eq. (5).

Equivalently, $\mathcal{Q}_\lambda^{-1}\mathcal{P}_\lambda\mathcal{Q}_\lambda=\mathcal{P}_\lambda+2\pi\hbar\lambda^2/L$. We may then compute the time-evolved large-scale position operator:

$$\begin{aligned}\mathcal{Q}_\lambda(\tau)&=\exp[it/(\hbar 2m)\mathcal{P}_\lambda^2]\,\mathcal{Q}_\lambda\exp[-it/(\hbar 2m)\mathcal{P}_\lambda^2]\\&=\mathcal{Q}_\lambda\exp[it/(\hbar 2m)\{\mathcal{Q}_\lambda^{-1}\mathcal{P}_\lambda\mathcal{Q}_\lambda\}^2]\exp[-it/(\hbar 2m)\mathcal{P}_\lambda^2]\\&=\exp[it(2\pi\hbar\lambda^2/L)^2/(\hbar 2m)]\,\mathcal{Q}_\lambda\exp[i2\pi\lambda^2 t/(mL)\mathcal{P}_\lambda]\\&=\exp[i(2\pi)^2\,\hbar\lambda^2\tau/(2mL^2)]\,\mathcal{Q}_\lambda\exp[i(2\pi/L)(\tau/m)\mathcal{P}_\lambda]\end{aligned}$$

where we have substituted $\lambda^2 t=\tau$. We note that, as in the previous cases, we may translate the initial state of the particle around the circle by the amount $\lambda^{-2}X_0$. Shifting this translation to the observables results in a change in the last factor in the expression for $\mathcal{Q}_\lambda(\tau)$ to $\exp[i(2\pi/L)\{X_0+(\tau/m)\mathcal{P}_\lambda\}]$.

In order to study the limit $\lambda\to 0$, it is convenient to make the following convention. Let $E_\lambda$ be the projection in $\mathcal{H}$ onto $\mathcal{H}_\lambda$. If the wavefunction of the particle is in the orthogonal complement of $\mathcal{H}_\lambda$, we say the particle *is not on the circle* and assign its large-scale position on the circle to be zero (i.e., not on the circle). In this way we may consider $\mathcal{Q}_\lambda(\tau)$ to act on $\mathcal{H}$ and replace the formula for $\mathcal{Q}_\lambda(\tau)$ by

$$\begin{aligned}\mathcal{Q}_\lambda(\tau)=\exp[i(2\pi)^2\,\hbar\lambda^2\tau/(2mL^2)]\exp[i(2\pi\lambda^2/L)q]\\\times\exp[i(2\pi/L)\{X_0+(\tau/m)\mathcal{P}_\lambda\}]E_\lambda\end{aligned}\tag{6}$$

where $q$ is the position operator on $\mathcal{H}$. It is now an easy matter to study the limit $\lambda\to 0$. The first two factors in the expression for $\mathcal{Q}_\lambda(\tau)$ converge strongly to the identity operator, as does $E_\lambda$. Now let $\psi\in\mathcal{H}_{\lambda_0}$. For $\lambda<\lambda_0$, $\psi\in\mathcal{H}_\lambda$ and furthermore if $|a|<(\lambda^{-2}-\lambda_0^{-2})L$, then $\exp[i(a/\hbar)\mathcal{P}_\lambda]\psi=\exp[i(a/\hbar)\mathcal{P}]\psi$, since $\mathcal{P}_\lambda$ generates shifts on $[-\lambda^{-2}L,\lambda^{-2}L]$ with periodic boundary conditions and $\mathcal{P}$ generates shifts on $R$. Hence $\mathcal{Q}_\lambda(\tau)\psi\to\exp[i(2\pi/L)\{X_0+(\tau/m)\mathcal{P}\}]\psi$. Such wavefunctions $\psi$, for all $\lambda_0$, are dense in $\mathcal{H}$. Thus $\mathcal{Q}_\lambda(\tau)$ converges strongly to $\exp[i(2\pi/L)\{X_0+(\tau/m)\mathcal{P}\}]$. As in the previous discussions, this expression leads to joint large-scale multiple-time position measurements given by classical particle trajectories on a circle of rescaled circumference $L$, starting at $X_0$ and moving with velocity $p/m$, where the probability density for $p$ is $|\hat{\psi}(p)|^2$.

Translation on the circle by the large-scale amount $X$ is given by $U_\lambda(\lambda^{-2}X)=\exp[i(\lambda^{-2}X/\hbar)\mathcal{P}_\lambda]$ on $\mathcal{H}_\lambda$. We extend this to $\mathcal{H}$ by the convention that if the particle is not on the circle, it is not translated. Thus on $\mathcal{H}$,

$$U_\lambda(\lambda^{-2}X)=\exp[i(\lambda^{-2}X/\hbar)\mathcal{P}_\lambda]E_\lambda+(1-E_\lambda)$$

Suppose $|X| < L/2$ and let the wavefunction $\psi$ have support in $[-K, K]$. Then

$$U_\lambda(\lambda^{-2}X)\psi = \exp[i(\lambda^{-2}X/\hbar)\mathscr{P}_\lambda]\psi = \exp[i(\lambda^{-2}X/\hbar)\mathscr{P}]\psi$$

if $K + \lambda^{-2}|X| < \lambda^{-2}L/2$. We now see that a coherent superposition of two compact support wavefunctions $\psi_1, \psi_2$ translated by different macroscopic amounts $X_1, X_2$, where $|X_j| < L/2$, $|X_1 - X_2| < L/2$, gives cross-terms which tend to zero as $\lambda \to 0$ by the weak convergence to zero discussed in the remark of Section 2. This then extends to all wavefunctions in $\mathscr{H}$. Hence a *mixture* of $\psi_1$ at $X_1$ and $\psi_2$ at $X_2$ results.

### 2.3. Free Evolution on an Interval

The method of images may again be used to obtain the time evolution of a quantum particle restricted to an interval $[0, \lambda^{-2}L/2]$ in terms of the evolution on a circle, which is given by periodic boundary conditions at the endpoint of $[-\lambda^{-2}L/2, \lambda^{-2}L/2]$. Let $\mathscr{H}_{\lambda,D} = L^2([0, \lambda^{-2}L/2], dx)$ and, as in the previous discussion of the half-line, consider the unitary map $A_\lambda$ of $\mathscr{H}_{\lambda,D}$ onto the odd subspace $\mathscr{H}_{\lambda,0}$ of $\mathscr{H}_\lambda = L^2([-\lambda^{-2}L/2, \lambda^{-2}L/2], dx)$ given by $(A_\lambda\psi)(x) = \mathrm{sign}(x)\,2^{-1/2}\psi(|x|)$ and the inverse map $B_\lambda$: $(B_\lambda\psi)(x) = 2^{1/2}\psi(x)$. The self-adjoint operator $H_{\lambda,D}$ on $\mathscr{H}_{\lambda,D}$ is defined by $\exp[itH_{\lambda,D}] = B_\lambda \exp[it/(\hbar 2m)\mathscr{P}_\lambda^2]A_\lambda$, where $\mathscr{P}_\lambda^2/(2m)$ is the Hamiltonian on the circle. A wavefunction in the domain of $\mathscr{P}_\lambda^2$ is necessarily continuous with continuous first derivative (and periodic at $\pm\lambda^{-2}L/2$) and hence if $\psi \in \mathscr{H}_{\lambda,0}$, it follows that $\psi$ is zero at $x = 0$ and furthermore, due to the periodic boundary conditions at the endpoints $\pm\lambda^{-2}L/2$, also at the endpoints. Thus $B_\lambda\psi \in \mathscr{H}_{\lambda,D}$ is zero at $x = 0$ and $x = \lambda^{-2}L/2$. In this way we see that $H_{\lambda,D}$ is $-(\hbar^2/2m)\,d^2/dx^2$ with Dirichlet boundary conditions at $x = 0$ and $x = \lambda^{-2}L/2$. (In a similar way the even subspace of $\mathscr{H}_\lambda$ gives rise to *Neumann* boundary conditions at $x = 0$ and $x = \lambda^{-2}L/2$.) The discussion now proceeds along similar lines to the case of the half-line. Thinking of the interval $[0, \lambda^{-2}L/2]$ as the semicircle of circumference $\lambda^{-2}L/2$ centered at the origin and contained in the upper-half complex plane, we take the rescaled space coordinate $\exp[i2\pi\lambda^2 x/L]$ (see Section 2.2). Then $\mathscr{Q}_{\lambda,D} = B_\lambda f(\mathscr{Q}_\lambda)A_\lambda$, where $f(z) = \Re z + i|\Im z|$ is continuous, and $\mathscr{Q}_{\lambda,D}(\tau) = B_\lambda f(\mathscr{Q}_\lambda(\tau))\,A_\lambda$. As in the previous section, if the wavefunction of the particle is the orthogonal complement of $\mathscr{H}_{\lambda,D}$ in $\mathscr{H}_D$ (Section 2.1), we say the particle is not in the interval and assign its position to be zero. As in Section 2.2, we extend $\mathscr{Q}_{\lambda,D}(\tau)$ to an operator on $\mathscr{H}_D$ by $\mathscr{Q}_{\lambda,D}(\tau) = B_\lambda f(\mathscr{Q}_\lambda(\tau))\,A_\lambda\mathscr{E}_\lambda$, where $\mathscr{E}_\lambda$ is the projection in $\mathscr{H}_D$ onto $\mathscr{H}_{\lambda,D}$. We may replace $A_\lambda$ by $A$, $B_\lambda$ by $B$ (Section 2.1), and take $\mathscr{Q}_\lambda(\tau)$ as in Eq. (6). It follows that $\mathcal{Q}_{\lambda, D}(\tau)$ converges strongly to $Bf(\exp[i(2\pi/L)(X_0+(\tau/m)\mathcal{P})])A$. As in previous discussions (Sections 2.1 and, this formula leads to classical trajectories on the interval. The presence of the function $f$ has the effect of reflecting the trajectories at the endpoints of the interval. (The presence or absence of cross-terms depending on whether the particle is "at $X=0$ or $X=L/2$" or at a macroscopically interior point of the interval follows a similar discussion to that in Section 2.1.)

**Remark.** The preceding one-dimensional constructions extend immediately to higher dimensions, using the fact that the Hamiltonian is a sum of mutually commuting and independent one-dimensional Hamiltonians, and thus each component of the large-scale position operator evolves as in the one-dimensional case. We may thus obtain the joint distribution for large-scale multiple-time position measurements in terms of classical geodesic trajectories on cylinders, tori, and hyper-rectangles.

## 2.4. Free Evolution on a Lattice

Associated with a quantum particle on a one-dimensional lattice with spacing $l$ is the Hilbert space of wavefunctions $\mathcal{H}=l^2(Z)$. The unitary shift operator $U$ is defined by $(U\psi)(n)=\psi(n-1)$ and the lattice Laplacian is $\Delta=l^{-2}[U+U^{-1}-2]$. The Hamiltonian is $H=-h^2/(2m)\Delta$. The position operator $q$ is given by $(q\psi)(n)=nl\psi(n)$ and $q(t)=\exp[(it/h)H]q\exp[-(it/h)H]$. Then $dq/dt=h/(ml)(i/2)[U-U^{-1}]=(1/m)\mathcal{P}$, where the momentum operator $\mathcal{P}=(h/l)(i/2)[U-U^{-1}]$. Note that $d\mathcal{P}/dt=0$, so the momentum is conserved and $q(t)=q+(t/m)\mathcal{P}$. Furthermore, if the initial wavefunction is shifted by $\lambda^{-2}X_0$ and this is transferred to the observables, then $q(t)=\lambda^{-2}X_0+q+(t/m)\mathcal{P}$. The large-scale position operator is

$$\mathcal{Q}(\tau)=\lambda^2 q(\lambda^{-2}\tau)=\lambda^2 q+X_0+(\tau/m)\mathcal{P}$$

The operator $\mathcal{P}$ is bounded. Indeed, Fourier transformation is defined by

$$\hat{\psi}(p)=\sum_n \psi(n)\exp[-inp]$$

$$\psi(n)=(2\pi)^{-1}\int_{-\pi}^{\pi} dp\,\hat{\psi}(p)\exp[inp]$$

and

$$\widehat{U\psi}(p)=\exp[-ip]\,\hat{\psi}(p)$$

Hence

$$\widehat{H\psi}(p) = h^2/(ml^2)[1 - \cos p]\,\hat{\psi}(p)$$
$$\widehat{\mathscr{P}\psi}(p) = (h/l)\sin p\hat{\psi}(p)$$

so $-(h/l) \leqslant \mathscr{P} \leqslant (h/l)$. Then $\mathscr{Q}(\tau)$ converges strongly on the domain of $q$ to the bounded operator $X_0 + (\tau/m)\mathscr{P}$. Hence, observed on a large space-time scale, the particle moves on a straight-line trajectory $X_0 + (\tau/m)(h/l)\sin p$ with probability density

$$(2\pi)^{-1}\,[|\hat{\psi}(p)|^2 + |\hat{\psi}(\text{sign}(p)\pi - p)|^2]\,dp, \qquad -\pi/2 \leqslant p \leqslant \pi/2$$

The maximum speed is $h/lm$.

**Remark.** It is interesting to note that the Bessel function identity

$$2nJ_n(t) = t[J_{n-1}(t) + J_{n+1}(t)] \tag{7}$$

is an immediate consequence of the operator identity $q(t) = q + (t/m)\mathscr{P}$. Indeed, taking $h = m = l = 1$, and denoting by $\delta_0$ the wavefunction which is 1 at $n = 0$ and 0 otherwise,

$$(\exp[itH]\,\delta_0)(n) = \exp[it](-i)^n\,J_n(t) \tag{8}$$

and since $q\delta_0 = 0$, we have

$$0 = q(t)\exp[itH]\,\delta_0 = [q + t(i/2)(U - U^{-1})]\exp[itH]\delta_0 \tag{9}$$

Equations (8) and (9) give the identity (7).

**Remark.** The kernel of the free evolution on the lattice is given by Bessel functions [Eq. (8)]. In the continuum $R$ the kernel is *uniformly* bounded by $Ct^{-1/2}$. A uniform bound $Ct^{-1/4}$ for the lattice kernel was obtained in ref. 12, Appendix C, but the best uniform bound has the form $Ct^{-1/3}$.[27]

## 2.5. Uniform Gravitational Field

A uniform large-scale gravitational field in $R^\nu$ is described in terms of the large-scale coordinate $X$ by the potential $V(X) = mgX$ and hence in terms of the coordinate $x = \lambda^{-2}X$, $V(x) = m\lambda^2 gx$. Thus in this case the interaction strength $\lambda^2 g$ is scaled in the same way as the space-time coordinates. The Heisenberg equations of motion associated with the Hamiltonian $H = 1/(2m)\mathscr{P}^2 + \lambda^2 mgq$ are $dq(t)/dt = (1/m)\mathscr{P}(t)$ and $d\mathscr{P}(t)/dt = -\lambda^2 mg$. Then $\mathscr{P}(t) = \mathscr{P} - \lambda^2 mgt$ and $q(t) = q + (t/m)\mathscr{P} -$

$\lambda^2 g t^2/2 + \lambda^{-2} X_0$, where we have included a large-scale displacement by $X_0$. Hence $\mathscr{P}(\tau) = \mathscr{P} - mg\tau$ and $\mathscr{Q}(\tau) = \lambda^2 q(\lambda^{-2}\tau) = \lambda^2 q + (\tau/m)\mathscr{P} - g\tau^2/2 + X_0$, which converges to $X_0 + (\tau/m)\mathscr{P} - g\tau^2/2$. In higher dimensions with the gravitational field in the $\hat{z}$ direction we have $\mathscr{P}(\tau) = \mathscr{P} - mg\tau\hat{z}$ and $\mathscr{Q}(\tau) = X_0 + (\tau/m)\mathscr{P} - g(\tau^2/2)\hat{z}$. *The classical trajectories are thus parabolas.*

## 2.6. Slowly Varying Potentials

A quantum particle subjected to a slowly varying potential (one which varies on the macroscopic scale of the form $V(X) = V(\lambda^2 x)$ should, in the large space-time limit $\lambda \to 0$, be described by trajectories $\mathscr{Q}(\tau) = F_\tau(X_0, \mathscr{P})$, where $X(\tau) = F_\tau(x, p)$ is the classical trajectory satisfying

$$\frac{dX(\tau)}{d\tau} = \frac{1}{m} p(\tau)$$

$$\frac{dp(\tau)}{d\tau} = \nabla V(X(\tau))$$

with initial conditions $X(0) = x$, $dX/d\tau(0) = p$. This was shown in Section 2.5 for a slowly varying gravitational potential. The calculation for the harmonic oscillator potential $V(X) = 1/2kX^2 = 1/2\lambda^4 k x^2$ is also easily done. The Heisenberg equations of motion are

$$\frac{dq(t)}{dt} = \frac{1}{m}\mathscr{P}(t)$$

$$\frac{d\mathscr{P}(t)}{dt} = -\lambda^4 k q(t)$$

or

$$\frac{d\mathscr{Q}(t)}{d\tau} = \frac{1}{m}\mathscr{P}(\tau)$$

$$\frac{d\mathscr{P}(\tau)}{d\tau} = -k\mathscr{Q}(\tau)$$

with solution

$$\mathscr{Q}(\tau) = \mathscr{Q}\cos(\omega\tau) + \mathscr{P}\frac{\sin(\omega\tau)}{m\omega}$$

Substituting $\mathcal{Q} = \lambda^2 q + X_0$ gives, in the limit $\lambda \to 0$,

$$\mathcal{Q}(\tau) = X_0 \cos(\omega\tau) + \mathcal{P} \frac{\sin(\omega\tau)}{m\omega} = F_\tau(X_0, \mathcal{P})$$

A related approach with a different scaling leading to classical behavior is given by Hepp.[11]

## 2.7. Large-Space-Time Observation of Scattering

The quantum mechanical scattering states of a particle in $R^\nu$ are those wavefunctions $\psi_\pm(\phi)$ which for large times behave as free particles:

$$\|\exp[-i(t/\hbar)H]\,\psi_\pm(\phi) - \exp[-i(t/\hbar)\,H_0]\,\phi\| \to 0 \tag{10}$$

as $t \to \pm\infty$, where $H$ is the total Hamiltonian and $H_0$ is the free Hamiltonian $1/(2m)\mathcal{P}^2$. The Møller wave operators $\Omega_\pm$ are defined by $\Omega_\pm \phi = \psi_\pm(\phi)$ and the $S$-matrix $S$ by $\Omega_- \phi = \Omega_+ S_\phi$. Note that $\Omega_\pm$ are isometric: $\|\Omega_\pm \phi\| = \|\phi\|$. From Eq. (10) $\exp[i(t/\hbar)H]\exp[-i(t/\hbar)H_0]$ converges strongly to $\Omega_\pm$ and $\exp[i(t/\hbar)\,H_0]\ \exp[-i(t/\hbar)H]\,\Omega_\pm$ converges strongly to the identity operator as $t \to \pm\infty$. Let

$$\mathcal{Q}_H(\tau) = \lambda^2 \exp[i(\lambda^{-2}\tau/\hbar)H]\,q\,\exp[-i(\lambda^{-2}\tau/\hbar)H]$$

denote the large-scale position operator evolved with respect to the total Hamiltonian and $\mathcal{Q}(\tau)$, as in preceding sections, evolved with respect to the free Hamiltonian. Let $F(x)$ be a bounded measurable function which is continuous except on a closed set of Lebesgue measure zero. Then on the scattering states, if $\tau < 0$, as $\lambda \to 0$

$$\begin{aligned} F(\mathcal{Q}_H(\tau))\Omega_- &= \exp[i\lambda^{-2}(\tau/\hbar)H]\,\exp[-i\lambda^{-2}(\tau/\hbar)H_0]\;F(\mathcal{Q}(\tau)) \\ &\quad \times \exp[i\lambda^{-2}(\tau/\hbar)H_0]\,\exp[-i\lambda^{-2}(\tau/\hbar)H]\,\Omega_- \\ &\to \Omega_- F((\tau/m)\mathcal{P}) \end{aligned}$$

It follows that if $\tau_1$ and $\tau_2$ are negative,

$$\begin{aligned} &F_1(\mathcal{Q}_H(\tau_1))\,F_2(\mathcal{Q}_H(\tau_2))\Omega_- \\ &\quad = F_1(\mathcal{Q}_H(\tau_1))[F_2(\mathcal{Q}_H(\tau_2))\Omega_- - \Omega_- F_2((\tau_2/m)\mathcal{P})] \\ &\qquad + F_1(\mathcal{Q}_H(\tau_1))\,\Omega_-\,F_2((\tau_2/m)\mathcal{P}) \\ &\quad \to \Omega_- F_1((\tau_1/m)\mathcal{P})\,F_2((\tau_2/m)\mathcal{P}) \\ &\quad = \Omega_+ S F_1((\tau_1/m)\mathcal{P})\,F_2((\tau_2/m)\mathcal{P}) \end{aligned}$$

and similarly for a product of any number of such functions $F_j$. In a similar manner we conclude that if $\tau_1,\dots,\tau_k$ are $<0$ and $\tau_{k+1},\dots,\tau_N$ are $>0$, then, as $\lambda\to 0$,

$$F_N(\mathcal{Q}_H(\tau_N))\cdots F_1(\mathcal{Q}_H(\tau_1))\Omega_- \to \Omega_+ F_N((\tau_N/m)\mathcal{P}) \cdots F_{k+1}((\tau_{k+1}/m)\mathcal{P})\, SF_k((\tau_k/m)\mathcal{P})\cdots F_1((\tau_1/m)\mathcal{P}) \tag{11}$$

Using Eq. (1) for large-scale space-time measurements, and writing the equation in the form

$$P_\psi(\Delta_1,\tau_1;\dots;\Delta_N,\tau_N)=\|\chi_{\Delta_N}(\mathcal{Q}_H(\tau_N))\cdots\chi_{\Delta_1}(\mathcal{Q}_H(\tau_1))\psi\|^2 \tag{12}$$

we take $\tau_1<\cdots<\tau_k<0<\tau_{k+1}<\cdots<\tau_N$ and obtain in the limit $\lambda\to 0$, from Eq. (11), for an incoming scattering state $\psi=\Omega_-\phi$,

$$\begin{aligned}P_{\Omega_-\phi}&(\Delta_1,\tau_1;\dots;\Delta_N,\tau_N)\\ &=\|\chi_{\Delta_N}((\tau_N/m)\mathcal{P})\cdots\chi_{\Delta_{k+1}}((\tau_{k+1}/m)\mathcal{P})\\ &\quad\times S\chi_{\Delta_k}((\tau_k/m)\mathcal{P})\cdots\chi_{\Delta_1}((\tau_1/m)\mathcal{P})\phi\|^2\end{aligned} \tag{13}$$

since $\Omega_+$ is isometric. We see that $P_{\Omega_-\phi}(\Delta_1,\tau_1;\dots;\Delta_N,\tau_N)=0$ unless $(\Delta_1,\tau_1),\dots,(\Delta_k,\tau_k)$ contain a straight-line trajectory $(\tau/m)p$ with $p$ in the support of $\phi$, and $(\Delta_{k+1},\tau_{k+1}),\dots,(\Delta_N,\tau_N)$ contain a straight-line trajectory $(\tau/m)p'$ with $p'$ in the support of $S\chi_{\Delta_k}((\tau_k/m)\mathcal{P})\cdots\chi_{\Delta_1}((\tau/m)\mathcal{P})\phi$. However, the expression (13) contains quantum mechanical interference terms and does not have the form given by the scattering of classical particles. In particular, summing over all $\Delta$ in the partition,

$$\sum_{\Delta_1,\dots,\Delta_k} P_{\Omega_-\phi}(\Delta_1,\tau_1;\dots;\Delta_N,\tau_N)\neq P_{\Omega_-\phi}(\Delta_{k+1},\tau_{k+1};\dots;\Delta_N,\tau_N)$$

If, however, the incoming state is sufficiently concentrated about the incoming momentum $p_0$ (in relation to the fineness of the partition), then at each time $\tau_j<0$ only the one $\Delta_j^0$ containing $(\tau_j/m)p_0$ will give a non-zero probability. (That is, to the accuracy of the partition, there is only one incoming trajectory $(\tau/m)p_0$. Furthermore,

$$\chi_{\Delta_k^0}((\tau_k/m)\mathcal{P})\cdots\chi_{\Delta_1^0}((\tau_1/m)\mathcal{P})\phi=\phi$$

and so the probability density of observing an outgoing trajectory $(\tau/m)p'$ is $|(S\phi)(p')|^2$. (In order to relate this to the standard cross-section formula, it is necessary to average the probability density over the wavefunctions $\phi$ in the incoming particle beam.[24])

## 2.8. Many Particles

The analysis leading to Eq. (3) carries over unchanged to the case of $N$ quantum particles moving in $R^\nu$. If the $N$-particle wavefunction is $\psi(x^{(1)},..., x^{(N)})$, then the probability of finding particles in macroscopic space-time regions is given by $N$ straight-line trajectories $X^{(j)}(\tau) = (\tau/m) p^{(j)}$ distributed according to the probability density $|\hat{\psi}(p^{(1)},..., p^{(N)})|^2$. If, furthermore, the $j^{th}$ quantum particle is translated by the macroscopic amount $X_0^{(j)}$, then the trajectories are $X^{(j)}(\tau) = X_0^{(j)} + (\tau/m)\, p^{(j)}$ with the above probability density. If the quantum particles are uncorrelated, the wavefunction factorizes, $\psi(x^{(1)},..., x^{(N)}) = \psi^{(1)}(x^{(1)}) \cdots \psi^{(N)}(x^{(N)})$, and consequently the joint probability density of the classical particle trajectories also factorizes. Thus the trajectories are uncorrelated.

A coherent superposition of wavefunctions translated by different macroscopic amounts $(X_0^{(1)},..., X_0^{(N)})$ and $(X_0^{(1)'},..., X_0^{(N)'})$ will, in the large-scale limit $\lambda \to 0$, lead to a *mixture* as in previous discussions due to the cross-terms tending to zero:

$$U^{(1)}(\lambda^{-2}[X_0^{(1)} - X_0^{(1)'}]) \otimes \cdots \otimes U^{(N)}(\lambda^{-2}[X_0^{(N)} - X_0^{(N)'}])$$

tends weakly to zero if at least one $X_0^{(j)} - X_0^{(j)'}$ is nonzero.

If the particles are bosons or fermions, the wavefunction must be symmetric or antisymmetric, respectively. If the particles are translated by different macroscopic amounts, then symmetrization or antisymmetrization leads to a mixture due to the cross-terms tending to zero. Furthermore, for properly symmetrized observables each permuted wavefunction will lead to the same joint probabilities. Consequently for particles initially at different macroscopic points, Bose or Fermi statistics do not lead to any modification in joint multiple-time position probabilities. In particular, this is the case if the particles have an initial distribution of macroscopic positions $X_0$ which is absolutely continuous with respect to Lebesgue measure (i.e., given by a density function), since the probability of two particles being at the same point is zero. For example, suppose the $j$th quantum particle is initially in the (mixed) state $S^{(j)} = \sum_k c_k^{(j)} S_{\phi_k^{(j)}}$, where $S_{\phi_k^{(j)}}$ is the pute state determined by the wavefunction $\phi_k^{(j)}$. Then the momentum distribution is given by the density $g^{(j)}(p) = \sum_k c_k^{(j)} |\hat{\phi}_k^{(j)}(p)|^2$. Suppose the initial distribution of the macroscopic position of the $j$th particle is given by the density $h^{(j)}(X_0^{(j)})$. If the particles are uncorrelated, then $S = \bigotimes_{j=1}^N S^{(j)}$ and the trajectories $X^{(j)}(\tau) = X_0^{(j)} + (\tau/m) p^{(j)}$ are uncorrelated.

A similar discussion applies to motion in the previously discussed geometrical regions. A quantum particle on a circle of circumference $L$ with momentum distribution given by the density $g(p)$ and initial macroscopic position distribution given by the density $h(X_0)$ has moments of the position operator at macroscopic time $\tau$, in the large-scale limit,

$$\langle U^n \rangle = \int dX_0 h(X_0) \int dp\, g(p) \exp[in(2\pi/L)(X_0 + (\tau/m)p)] \qquad (14)$$

Denote by $\mu_\tau$ the probability measure on the unit circle determined by these moments:

$$\langle U^n \rangle = \int_0^{2\pi} d\mu_\tau(\theta) \exp[in\theta]$$

The moments [Eq. (14)] for $n \neq 0$ converge to zero as $\tau \to \infty$ by the Riemann–Lebesgue lemma. These limiting values are equal to $\int_0^{2\pi} d\mu(\theta) \exp[in\theta]$, where $\mu$ is the uniform distribution $d\mu = (2\pi)^{-1}\, d\theta$. It follows[9] that the probability $\mu_\tau(\alpha)$ of finding the particle on any arc $\alpha$ of length $|\alpha|$ on the circle converges to $\mu(\alpha) = |\alpha|/(2\pi)$. If we now consider $N$ uncorrelated quantum particles, each in any state $S^{(j)}$ with distribution $h^{(j)}(X_0^{(j)})$, then in the large-scale limit, as $\tau \to \infty$ and for sufficiently large $N$, the density of particles will exhibit very small fluctuations around the uniform density. (The discussion can be further extended to the case where $g$ depends on both $p$ and $X_0$.)

## 2.9. A Quantum Counting Process

Here we discuss an alternative approach to position measurements on a free quantum particle which corresponds to setting up counters in various regions $\Delta_j$, $j = 1,\ldots, J$, and observing the times at which each counter clicks. A good exposition of the general approach developed by E. B. Davies may be found in the article by Srinivas and Davies.[21] We shall show that in a large-space-time limit, the quantum counting process goes over to the classical counting process associated with particles moving on the straight-line trajectories $X_0 + (\tau/m)p$.

To the counter in the region $\Delta_j$ is associated a counting rate "super-operator" $\mathscr{I}_{\Delta_j}$ acting on density matrices $\rho$. We shall take $\mathscr{I}_{\Delta_j}$ of the form

$$\mathscr{I}_{\Delta_j} \rho = \sigma^2 \chi_{\Delta_j}(q)\, \rho \chi_{\Delta_j}(q)$$

[9] The continuous functions $\exp[in\theta]$ are closed under multiplication and complex conjugation and separate points of the circle. By the Stone–Weierstrass theorem, finite linear combinations of such functions are dense in the supremum norm in the set of all continuous functions on the circle. Thus $\mu_\tau \to \mu$ weakly, and hence $\mu_\tau(B) \to \mu(B)$ for all Borel sets satisfying $\mu(\partial B) = 0$ (ref. 23, Theorem 1.1.1), in particular for $B$ an arc.

The total rate operator is

$$R = \sum_{j=1}^{J} \mathscr{J}^*_{\Delta_j}(I) = \sigma^2 \sum_{j=1}^{J} \chi_{\Delta_j}(q)$$

(where $\mathscr{J}^*_{\Delta_j}$ is the dual of $\mathscr{J}_{\Delta_j}$, acting on the observables).

In the absence of counters the density matrix evolves according to the Hamiltonian evolution

$$\exp[-i(t/\hbar)H]\,\rho\,\exp[i(t/\hbar)H]$$

where $H = 1/(2m)\mathscr{P}^2$. In the presence of the counters the evolution of the density matrix to time $t$ in the case that no counters click in the time interval $[0, t)$ is given by $S_t\rho/\mathrm{Tr}(S_t\rho]$, and the probability for this case to occur is $\mathrm{Tr}[S_t\rho]$. The "superoperator" $S_t$ acts on $\rho$ by

$$S_t\rho = \exp[-tK]\,\rho\,\exp[-tK^*]$$

where

$$K = (i/\hbar)H + R/2$$

$$K^* = -(i/\hbar)H + R/2$$

If $0 \leqslant t_1 \leqslant \cdots \leqslant t_l \leqslant t$, the probability density that the counter in $\Delta_{j_1}$ clicks at time $t_1$,..., and the counter in $\Delta_{j_l}$ clicks at time $t_l$, and no counter clicks in the rest of the interval $[0, t)$ is

$$P_\rho(t_1, \Delta_{j_1};...; t_l, \Delta_{j_l}) = \mathrm{Tr}[S_{t-t_l}\mathscr{J}_{\Delta_{j_l}}S_{t_l-t_{l-1}}\mathscr{J}_{\Delta_{j_{l-1}}}\cdots\mathscr{J}_{\Delta_{j_2}}S_{t_2-t_1}\mathscr{J}_{\Delta_{j_1}}S_{t_1}\rho] \tag{15}$$

If the state of the particle is given by the wavefunction $\psi$, the density matrix $\rho$ is the projection onto $\psi$,

$$\rho = |\psi\rangle\langle\psi|$$

in the Dirac bra-ket notation. In this case

$$S_t\,|\psi\rangle\langle\psi| = |\psi_t\rangle\langle\psi_t|$$

where

$$\psi_t = \exp[-tK]\,\psi$$

and

$$\mathscr{J}_{\Delta_j}\,|\psi\rangle\langle\psi| = |\psi_{\Delta_j}\rangle\langle\psi_{\Delta_j}|$$

where

$$\psi_{\Delta_j} = \sigma \chi_{\Delta_j}(q)\psi$$

Consequently, the probability density (15) is then

$$P_\psi(t_1, \Delta_{j_1};...; t_l, \Delta_{j_l}) = \|\psi'\|^2$$

where

$$\psi' = \sigma^l \exp[-(t-t_l)K]\, \chi_{\Delta_{j_l}}(q) \exp[-(t_l - t_{l-1})K] \cdots$$
$$\times \exp[-(t_2 - t_1)K]\, \chi_{\Delta_{j_1}}(q) \exp[-t_1 K]\psi$$

We shall develop a macroscopic (large-space-time) form of the above analysis by replacing the counting rate superoperator $\mathscr{J}_{\Delta_j}$ by $\mathscr{J}^\lambda_{\Delta_j}$, where

$$\mathscr{J}^\lambda_{\Delta_j}\rho = \lambda^2\sigma^2\chi_{\Delta_j}(Q)\, \rho\chi_{\Delta_j}(Q)$$

where $Q = \lambda^2 q + X_0$, in which case the total rate operator $R$ is replaced by $R^\lambda$, where

$$R^\lambda = \lambda^2\sigma^2 \sum_{j=1}^{J} \chi_{\Delta_j}(Q)$$

The probability density with respect to the rescaled counting times $\tau_1,..., \tau_l$ is then

$$P_\psi(\tau_1, \Delta_{j_1};...; \tau_l, \Delta_{j_l}; [0, \tau)) = \|\psi_\lambda\|^2$$

where

$$\psi_\lambda = \sigma^l \exp[i\lambda^{-2}\tau/\hbar H] \exp[-\lambda^{-2}(\tau - \tau_l)K]$$
$$\times \chi_{\Delta_{j_l}}(Q) \exp[-\lambda^{-2}(\tau_l - \tau_{l-1})K] \cdots$$
$$\times \chi_{\Delta_{j_2}}(Q) \exp[-\lambda^{-2}(\tau_2 - \tau_1)K]\, \chi_{\Delta_{j_1}}(Q) \exp[-\lambda^{-2}\tau_1 K]\psi$$

The wavefunction $\psi_\lambda$ converges in $\mathscr{H}$ as $\lambda \to 0$ to $\psi_0$, where $\psi_0$ may be computed in the following way. An expression such as

$$\exp[\lambda^{-2}\tau_k K]\chi_{\Delta_{j_k}}(Q) \exp[-\lambda^{-2}\tau_k K]$$

is written in the form

$$W^\lambda_k \chi_{\Delta_{j_k}}(Q(\tau_k))(W^\lambda_k)^{-1}$$

where

$$W^\lambda_k = \exp[\lambda^{-2}\tau_k K] \exp[-i\lambda^{-2}\tau_k/\hbar H]$$

Now $\chi_{\Delta_{j_k}}(Q(\tau_k))$ converges strongly to $\chi_{\Delta_{j_k}}(X_0+(\tau_k/m)\mathscr{P})$ as $\lambda \to 0$. Furthermore, $W_k^\lambda$ converges strongly to

$$\exp\left[2^{-1}\sigma^2 \sum_{j=1}^{J} \int_0^{\tau_k} ds\, \chi_{\Delta_{j_k}}(X_0+(s/m)\mathscr{P})\right] \tag{16}$$

In order to prove (16), expand $W_k^\lambda$ in the norm convergent series

$$W_k^\lambda = \sum_{n=0}^{\infty} 2^{-n} \int_0^{\tau_k} ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\, \mathscr{R}_\lambda(s_n) \cdots \mathscr{R}_\lambda(s_1) \tag{17}$$

where $\mathscr{R}_\lambda(s) = \sigma^2 \sum_{j=1}^{J} \chi_{\Delta_j}(Q(s))$. Now $\mathscr{R}_\lambda(s)$ converges strongly as $\lambda \to 0$ to $\mathscr{R}_0(s) = \sigma^2 \sum_{j=1}^{J} \chi_{\Delta_j}(X_0+(s/m)\mathscr{P})$. Furthermore, $\|R_\lambda(s)\| \leqslant \sigma^2$ for all $\lambda$ and $s$, which implies that the series expansion (17) converges uniformly in $\lambda$. Hence we may interchange the sum over $n$ with the limit $\lambda \to 0$. The commutativity of $\mathscr{R}_0(s)$ for all $s$ then leads to (16). Similarly

$$(W_k^\lambda)^{-1} = \exp[i\lambda^{-2}\tau_k/\hbar H] \exp[-\lambda^{-2}\tau_k K]$$

converges strongly to

$$\exp\left[-2^{-1}\sigma^2 \sum_{j=1}^{J} \int_0^{\tau_k} ds\, \chi_{\Delta_{j_k}}(X_0+(s/m)\mathscr{P})\right]$$

Taking into account the commutativity of all factors in $\psi_0$, we obtain the expression

$$\psi_0 = \sigma^l \exp\left[-2^{-1}\sigma^2 \sum_{j=1}^{J} \int_0^{\tau} ds\, \chi_{\Delta_j}(X_0+(s/m)\mathscr{P})\right]$$
$$\times \chi_{\Delta_{j_l}}(X_0+(\tau_l/m)\mathscr{P}) \cdots \chi_{\Delta_{j_1}}(X_0+(\tau_1/m)\mathscr{P})\psi$$

Hence in the limit $\lambda \to 0$, $P_\psi(\tau_1, \Delta_{j_1};...; \tau_l, \Delta_{j_l}; [0, \tau))$ converges to

$$\int dp\, |\hat{\psi}(p)|^2 \sigma^{2l} \exp\left[-\sigma^2 \sum_{j=1}^{J} \int_0^{\tau} ds\, \chi_{\Delta_j}(X_0+(s/m)p\right]$$

$$\times \chi_{\Delta_{j_l}}(X_0+(\tau_l/m)p) \cdots \chi_{\Delta_{j_1}}(X_0+(\tau_1/m)p)$$

which are the corresponding classical counting probability densities for a particle moving on a straight-line trajectory $X_0+(\tau/m)p$ with probability density $|\hat{\psi}(p)|^2$.

## 3. A QUANTUM PARTICLE IN A RANDOM POTENTIAL

The large-space-time, weak-interaction limit is taken for a quantum particle moving in $R^v(v \geqslant 3)$ in the presence of a random field $v(x)$. The rescaled time and space variables are $\tau = \lambda^2 t$ and $\mathcal{Q} = \lambda^2 q$, and the interaction strength is $\lambda$. Formula (2) shall be used for the large-scale multiple-time position measurements with $\tau$ replacing $t$ and $\mathcal{Q}$ replacing $q$, and the limit $\lambda \to 0$ will be taken using the Dyson perturbation series—which may be controlled for small $\tau$ and Gaussian random fields $v(x)$. In place of the straight-line trajectories of Section 2, the trajectories describe a particle moving with constant speed in straight lines between random jumps in direction. In the limit $\lambda \to 0$ the joint distribution for multiple-time position measurements is thus described by a classical stochastic process.

With probability 1 with respect to the random field, the stochastic process obtained does not depend on the particular sample random field, but only on the ensemble value of the covariance of the random fields.

In this section, for notational simplicity, explicit formulas will be develop for $P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)$. The extension to more general multiple-time position measurements leads to expressions of the same general form.

### 3.1. Dyson Perturbation Series

**A.** Consider a Hamiltonian $H = H_0 + \lambda V$, where $H_0 = 1/(2m)\mathscr{P}^2$ and $V$ will for the moment be supposed bounded. The unitary operators

$$W(t) = \exp[(it/h)H] \exp[-(it/h)H_0]$$

satisfy

$$dW(t)/dt = (i\lambda/h)\; W(t)\; V(t)$$

where

$$V(t) = \exp[(it/h)H_0]\, V \exp[-(it/h)H_0]$$

Given a bounded operator $A$, $W(t)\, A\, W(t)^{-1}$ satisfies

$$d/dt[\, W(t)\, A\, W(t)^{-1}] = (i\lambda/h)\; W(t)[\, V(t), A]\; W(t)^{-1}$$

and hence

$$W(t)\, A\, W(t)^{-1} = A + (i\lambda/h) \int_0^t ds\; W(s)[\, V(s), A]\; W(s)^{-1}$$

Iterating this gives the norm-convergent Dyson perturbation series

$$W(t)\,A\,W(t)^{-1}=\sum_{n=0}^{\infty}(i\lambda/\hbar)^n\int_0^t ds_1\int_0^{s_1}ds_2\cdots$$
$$\times\int_0^{s_{n-1}}ds_n[V(s_n),...,[V(s_1),A]\cdots]$$

Expanding the commutators $[V(s_n),...,[V(s_1),A]\cdots]$ yields $2^n$ terms of the form

$$(-1)^{n-r}\,V(s'_1)\cdots V(s'_r)\,A\,V(s'_{r+1})\cdots V(s'_n)$$

where $s'_1,..., s'_n$ is a permutation of $s_1,..., s_n$. Let $\Phi_n$ denote the set of these $2^n$ *configurations*, a configuration being denoted by $\phi$ (ref. 12, § 5.2). Then $\phi$ determines the permutation $s'_1,..., s'_n$, which will now be denoted $s^\phi_1,..., s^\phi_n$. Also $r$ depends on $\phi$, but we shall not make this dependence explicit. Now denoting

$$A_H(t)=\exp[(it/\hbar H]\,A\exp[-(it/\hbar)H]$$
$$A(t)=\exp[(it/\hbar)H_0]A\exp[-(it/\hbar)H_0]$$

we have $A_H(t)=W(t)\,A(t)\,W(t)^{-1}$ and hence

$$A_H(\lambda^{-2}\tau)=\sum_{n=0}^{\infty}(i\lambda/\hbar)^n\sum_{\phi\in\Phi_n}(-1)^{n-r}\int_0^{\lambda^{-2}\tau}ds_1\cdots\int_0^{s_{n-1}}ds_n$$
$$\times V(s^\phi_1)\cdots V(s^\phi_r)\,A(\lambda^{-2}\tau)\,V(s^\phi_{r+1})\cdots V(s^\phi_n) \qquad (18)$$

It is notationally convenient to relabel the times $s_1,..., s_n$ at $t_1,..., t_n$, where $t_j=s^\phi_j$.[10] We may then reexpress (18) as

$$A_H(\lambda^{-2}\tau)=\sum_{n=0}^{\infty}(-i\lambda/\hbar)^n\sum_{\phi\in\Phi_n}(-1)^r\int dt_1\cdots dt_n\,V(t_1)\cdots V(t_r)$$
$$\times A(\lambda^{-2}\tau)\,V(t_{r+1})\cdots V(t_n) \qquad (19)$$

where $\int dt_1\cdots dt_n$ denotes integration over the appropriate sector of $[0,\lambda^{-2}\tau]^n$.

**B.** If $A$ and $B$ are bounded operators and $0\leqslant\tau_1\leqslant\tau_2$, then

$$A_H(\lambda^{-2}\tau_1)\,B_H(\lambda^{-2}\tau_2)\,A_H(\lambda^{-2}\tau_1)=\{AB_H(\lambda^{-2}(\tau_2-\tau_1))\,A\}_H(\lambda^{-2}\tau_1)$$

[10] Notice that $t_1,..., t_n$ are totally ordered, but the ordering is not given by the index $j$ of $t_j$.

Applying Eq. (18) twice gives

$$
\begin{aligned}
&A_H(\lambda^{-2}\tau_1)\, B_H(\lambda^{-2}\tau_2\, A_H(\lambda^{-2}\tau_1)\\
&\quad=\sum_{n,n'} (i\lambda/\hbar)^{n+n'} \sum_{\phi\in\Phi_n} (-1)^{n-r} \sum_{\phi'\in\Phi_{n'}} (-1)^{n'-r'}\\
&\quad\times\int_0^{\lambda^{-2}\tau_1} du_1 \cdots \int_0^{u_{n-1}} du_n \int_{\lambda^{-2}\tau_1}^{\lambda^{-2}\tau_2} ds_1 \cdots \int_{\lambda^{-2}\tau_1}^{s_{n'-1}} ds_{n'}\\
&\quad\times V(u_1^\phi)\cdots V(u_r^\phi)\, A(\lambda^{-2}\tau_1)\\
&\quad\times \{V(s_1^{\phi'})\cdots V(s_{r'}^{\phi'})\, B(\lambda^{-2}\tau_2)\, V(s_{r'+1}^{\phi'})\cdots V(s_{n'}^{\phi'})\}\\
&\quad\times A(\lambda^{-2}\tau_1)\, V(u_{r+1}^\phi)\cdots V(u_n^\phi) &&(20)\\
&\quad=\sum_{N=0}^{\infty} (-i\lambda/\hbar)^N \sum_{n+n'=N} \sum_{\phi\in\Phi_n} \sum_{\phi'\in\Phi_{n'}} (-1)^{r+r'}\\
&\quad\times\int dt_1\cdots dt_N\, V(t_1)\cdots V(t_r)\, A(\lambda^{-2}\tau_1)\\
&\quad\times V(t_{r+1})\cdots V(t_{r+r'})\, B(\lambda^{-2}\tau_2)\, V(t_{r+r'+1})\cdots\\
&\quad\times V(t_{r+n'})\, A(\lambda^{-2}\tau_1)\, V(t_{r+n'+1})\cdots V(t_N) &&(21)
\end{aligned}
$$

where we have introduced $t_1,\ldots, t_N$ as in Part A.

In a similar way, the perturbative expansion is obtained for

$$A_H^{(1)}(\lambda^{-2}\tau_1)\cdots A_H^{(N)}(\lambda^{-2}\tau_N)\, A_H^{(N)}(\lambda^{-2}\tau_N)\cdots A_H^{(1)}(\lambda^{-2}\tau_1)$$

where $0\leqslant\tau_1\leqslant\cdots\leqslant\tau_N$.

## 3.2. The Random Field

**A.** Let the (nonrandom) field $v(x)$ be "well-behaved": $v(\mathbf{x})$ is the Fourier transform of the integrable function $\hat{v}(p)$ [and hence $v(x)$ is bounded]. The potential $V=v(q)$ is given by $(V\psi)(x)=v(x)\,\psi(x)$. We may write $V=v(q)=\int dk\, \hat{v}(k)\exp[ik\cdot q]$ and hence

$$V(t)=\int dk\, \hat{v}(k)\exp[ik\cdot q(t)] \tag{22}$$

Furthermore, according to formula (2), we shall be interested in observables $A$ of the form

$$A=\eta_\Delta(\mathscr{Q})=\eta_\Delta(\lambda^2 q)=\int dk\, \widehat{\eta_\Delta}(k)\exp[i\lambda^2 k\cdot q]$$

and hence

$$A(t)=\int dk\, \widehat{\eta_{\Delta}}(k) \exp[i\lambda^2 k\cdot q(t)] \tag{23}$$

According to formula (2), the probability that the particle is found in the (large-scale) region $\Delta_1$ at (rescaled) time $\tau_1$ and in $\Delta_2$ at time $\tau_2$, $0\leqslant\tau_1\leqslant\tau_2$, is given by

$$P_\psi(\Delta_1,\tau_1;\Delta_2,\tau_2)=(\psi, \eta_{\Delta_1}(\mathcal{Q}_H(\tau_1))\, \eta_{\Delta_2}(\mathcal{Q}_H(\tau_2))\,\eta_{\Delta_2}(\mathcal{Q}_H(\tau_2))\, \eta_{\Delta_1}(\mathcal{Q}_H(\tau_1))\psi) \tag{24}$$

Substituting Eq. (22) and (23) into Eq. (20), we have

$$\begin{aligned}
&\eta_{\Delta_1}(\mathcal{Q}_H(\tau_1))\, \eta_{\Delta_2}(\mathcal{Q}_H(\tau_2))\, \eta_{\Delta_2}(\mathcal{Q}_H(\tau_2))\, \eta_{\Delta_1}(\mathcal{Q}_H(\tau_1))\\
&\quad=\eta_{\Delta_1}(\mathcal{Q})_H(\lambda^{-2}\tau_1)\{\eta_{\Delta_2}(\mathcal{Q})\,\eta_{\Delta_2}(\mathcal{Q})\}_H(\lambda^{-2}\tau_2)\,\eta_{\Delta_1}(\mathcal{Q})_H(\lambda^{-2}\tau_1)\\
&\quad=\sum_{n,n'}(i\lambda/\hbar)^{n+n'}\sum_{\phi\in\Phi_n}\sum_{\phi'\in\Phi_{n'}}(-1)^{n-r+n'-r'}\\
&\qquad\times\int_0^{\lambda^{-2}\tau_1}du_1\cdots\int_0^{u_{n-1}}du_n\int_{\lambda^{-2}\tau_1}^{\lambda^{-2}\tau_2}ds_1\cdots\int_{\lambda^{-2}\tau_1}^{s_{n'-1}}ds_{n'} \qquad (25)\\
&\qquad\times\int dp_1\cdots dp_n\,dp'_1\cdots dp'_{n'}\,d\kappa_1\,d\kappa'_1\,d\kappa_2\,d\kappa'_2\\
&\qquad\times\hat{v}(p)\cdots\hat{v}(p_n)\,\hat{v}(p'_1)\cdots\hat{v}(p'_{n'})\,\widehat{\eta_{\Delta_1}}(\kappa_1)\,\widehat{\eta_{\Delta_1}}(\kappa'_1)\,\widehat{\eta_{\Delta_2}}(\kappa_2)\,\widehat{\eta_{\Delta_2}}(\kappa'_2)\\
&\qquad\times\exp[ip_1\cdot q(u_1^\phi)]\cdots\exp[ip_r\cdot q(u_r^\phi)]\exp[i\lambda^2\kappa_1\cdot q(\lambda^{-2}\tau_1)]\\
&\qquad\times\exp[ip'_1\cdot q(s_1^{\phi'})]\cdots\exp[ip'_{r'}\cdot q(s_{r'}^{\phi'})]\\
&\qquad\times\exp[i\lambda^2\kappa_2\cdot q(\lambda^{-2}\tau_2)]\exp[i\lambda^2\kappa'_2\cdot q(\lambda^{-2}\tau_2)]\\
&\qquad\times\exp[ip'_{r'+1}\cdot q(s_{r'+1}^{\phi'})]\cdots\exp[ip'_{n'}\cdot q(s_{n'}^{\phi'})]\\
&\qquad\times\exp[i\lambda^2\kappa'_1\cdot q(\lambda^{-2}\tau_1)]\exp[ip_{r+1}\cdot q(u_{r+1}^\phi)]\cdots\exp[ip_n\cdot q(u_n^\phi)]\\
&\quad=\sum_{N=0}^{\infty}(-i\lambda/\hbar)^N\sum_{n+n'=N}\sum_{\phi\in\Phi_n}\sum_{\phi'\in\Phi_{n'}}(-1)^{r+r'}\\
&\qquad\times\int dt_1\cdots dt_N\int dk_1\cdots dk_N\,d\kappa_1\cdots d\kappa_4 \qquad (26)\\
&\qquad\times\hat{v}(k_1)\cdots\hat{v}(k_N)\,\widehat{\eta_{\Delta_1}}(\kappa_1)\,\widehat{\eta_{\Delta_2}}(\kappa_2)\,\widehat{\eta_{\Delta_2}}(\kappa_3)\,\widehat{\eta_{\Delta_1}}(\kappa_4)\\
&\qquad\times\exp[ik_1\cdot q(t_1)]\cdots\exp[ik_r\cdot q(t_r)]\\
&\qquad\times\exp[i\lambda^2\kappa_1\cdot q(\lambda^{-2}\tau_1)]\exp[ik_{r+1}\cdot q(t_{r+1})]\cdots \qquad (27)
\end{aligned}$$

$$\times \exp[ik_{r+r'} \cdot q(t_{r+r'})] \exp[i\lambda^2 \kappa_2 \cdot q(\lambda^{-2}\tau_2)]$$

$$\times \exp[i\lambda^2\kappa_3 \cdot q(\lambda^{-2}\tau_2)] \exp[ik_{r+r'+1} \cdot q(t_{r+r'+1})] \cdots \tag{28}$$

$$\times \exp[ik_{r+n'} \cdot q(t_{r+n'})] \exp[i\lambda^2\kappa_4 \cdot q(\lambda^{-2}\tau_1)]$$

$$\times \exp[ik_{r+n'+1} \cdot q(t_{r+n'+1})] \cdots \exp[ik_N \cdot q(t_N)] \tag{29}$$

where in (26) we have substituted the variables $t_1,..., t_N$ for $u_1,...,\boldsymbol{u}_n, s_1,..., s_{n'}$ as in Section 3.1.

A similar formula holds for position measurements at any number of times $\tau$.

**B.** The product of exponentials contained in (27)–(29) can be simplified by repeatedly using the Campbell–Baker–Hausdorff formula $\exp A \exp B = \exp(A+B) \exp(1/2[A, B])$, which holds when $[A, B]$ commutes with both $A$ and $B$. Using $[q, \mathscr{P}] = i\hbar$ and $q(t) = q + (t/m)\mathscr{P}$ yields (ref. 12, § 5.5)

$$\begin{aligned} &\exp[ik_1 \cdot q(t_1)] \cdots \exp[ik_n \cdot q(t_n)] \\ &\quad = \exp[-(i\hbar/(2m)\mathscr{K}] \exp[(i/m)(t_1k_1 + \cdots + t_nk_n) \cdot \mathscr{P}] \\ &\qquad \times \exp[i(k_1 + \cdots + k_n) \cdot q] \\ &\quad = \exp[(i\hbar/(2m)\mathscr{K}'] \exp[i(k_1 + \cdots + k_n) \cdot q] \\ &\qquad \times \exp[(i/m)(t_1k_1 + \cdots + t_nk_n) \cdot \mathscr{P}] \end{aligned} \tag{30}$$

where the quadratic forms $\mathscr{K}$ and $\mathscr{K}'$ are given by

$$\begin{aligned} \mathscr{K} &= \sum_{j,l} t_{j \vee l} k_j \cdot k_l \\ \mathscr{K}' &= \sum_{j,l} t_{j \wedge l} k_j \cdot k_l \end{aligned} \tag{31}$$

and $j \vee l = \max\{j, l\}$, $j \wedge l = \min\{j, l\}$. Reexpressing the product of exponentials in (27)–(29) using (30) and (31) leads to the following formula for $P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)$, Eq. (24):

$$\begin{aligned} &(\psi, \eta_{\Delta_1}(\mathscr{Q}_H(\tau_1))\, \eta_{\Delta_2}(\mathscr{Q}_H(\tau_2))\, \eta_{\Delta_2}(\mathscr{Q}_H(\tau_2))\, \eta_{\Delta_1}(\mathscr{Q}_H(\tau_1))\psi) \\ &\quad = \sum_{N=0}^{\infty} (-i\lambda/\hbar)^N \sum_{n+n'=N} \sum_{\phi \in \Phi_n} \sum_{\phi' \in \Phi_{n'}} (-1)^{r+r'} \\ &\qquad \times \int dt_1 \cdots dt_N \int dk_1 \cdots dk_N\, d\kappa_1 \cdots d\kappa_4 \end{aligned}$$

$$\times \hat{v}(k_1) \cdots \hat{v}(k_N) \widehat{\eta_{\Delta_1}}(\kappa_1) \widehat{\eta_{\Delta_2}}(\kappa_2) \widehat{\eta_{\Delta_2}}(\kappa_3) \widehat{\eta_{\Delta_1}}(\kappa_4)$$
$$\times \exp[-(i\hbar/2m)\{\mathscr{K} + \mathscr{K}_1 + \mathscr{K}_2\}]$$
$$\times \left(\psi, \exp\left[(i/m)\left(\sum_{j=1}^{N} t_j k_j + \sum_{a=1}^{4} \tau_a \kappa_a\right) \cdot \mathscr{P}\right]\right.$$
$$\left. \times \exp\left[i\left(\sum_{j=1}^{N} k_j + \lambda^2 \sum_{a=1}^{4} \kappa_a\right) \cdot q\right] \psi\right) \tag{32}$$

where we have set $\tau_3 = \tau_2$ and $\tau_4 = \tau_1$. The quadratic form $\mathscr{K}$ depends only on $k_1, \ldots, k_N$ and is given by

$$\mathscr{K} = \sum_{j,l=1}^{N} t_{j \vee l} k_j \cdot k_l \tag{33}$$

The quadratic form $\mathscr{K}_1$ depends only on $\kappa_1, \ldots, \kappa_4$ and is given by

$$\mathscr{K}_1 = \lambda^2 \sum_{a,b=1}^{4} \tau_{a \vee b} \kappa_a \cdot \kappa_b \tag{34}$$

The quadratic form $\mathscr{K}_2$ contains the cross-terms between the $k$'s and the $\kappa$'s and is given by

$$\mathscr{K}_2 = 2 \sum_{a=1}^{4} \kappa_a \cdot \left[\tau_a \sum_{j=1}^{j(a)} k_j + \lambda^2 \sum_{j=j(a)+1}^{N} t_j k_j\right] \tag{35}$$

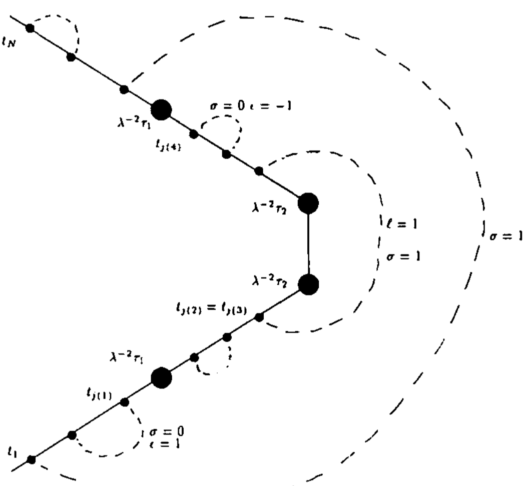


Fig. 1. A consecutive term with rescaled times $\tau_1, \tau_2$. Times increase to the right.

where we have set $j(1)=r$, $j(2)=r+r'$, $j(3)=r+r'$, $j(4)=r+n'$. (See Fig. 1)

A similar formula holds for general multiple-time position measurements.

**C.** Let $\mathscr{H}_1$ be the subset of wavefunctions $\psi$ such that $\psi(x)$ and its Fourier transform $\hat{\psi}(p)$ are integrable.[11] For $\psi_1, \psi_2 \in \mathscr{H}_1$ we may express the scalar product $(\psi_1, \psi_2)$ in the form[12]

$$\int dp\, dx \exp[-ip \cdot x/\hbar]\, \overline{\hat{\psi}_1(p)}\, \psi_2(x)$$

and hence, for $\psi \in H_1$,

$$\begin{aligned}(\psi, &\exp[ia \cdot \mathscr{P}] \exp[ib \cdot q] \psi)\\ &= \int dp\, dx \exp[-ip \cdot x/\hbar]\, \overline{\hat{\psi}(p)}\, \psi(x) \exp[ip \cdot a] \exp[ix \cdot b]\end{aligned}$$

Applying this expression to (32) gives the formula

$$\begin{aligned}P_\psi(&\Delta_1, \tau_1; \Delta_2, \tau_2)\\ &= \sum_{N=0}^{\infty} (-i\lambda/\hbar)^N \sum_{n+n'=N} \sum_{\phi \in \Phi_n} \sum_{\phi' \in \Phi_{n'}} (-1)^{r+r'}\\ &\quad \times \int dt_1 \cdots dt_N\, d\kappa_1 \cdots d\kappa_4\, \widehat{\eta_{\Delta_1}}(\kappa_1)\, \widehat{\eta_{\Delta_2}}(\kappa_2)\, \widehat{\eta_{\Delta_2}}(\kappa_3)\, \widehat{\eta_{\Delta_1}}(\kappa_4)\\ &\quad \times \int dp\, dx \exp[-ip \cdot x/\hbar]\, \overline{\hat{\psi}(p)}\, \psi(x)\\ &\quad \times \exp\left[i \sum_{a=1}^{4} (\lambda^2 x + (\tau_a/m)p) \cdot \kappa_a\right] \exp[-(i\hbar/2m)\mathscr{K}_1]\\ &\quad \times \int dk_1 \cdots dk_N\, \hat{v}(k_1) \cdots \hat{v}(k_N) \exp[-(i\hbar/2m)\mathscr{K}]\\ &\quad \times \exp i\left[\sum_{j=1}^{N} (x + (t_j/m)p) \cdot k_j - (\hbar/2m)\mathscr{K}_2\right]\end{aligned} \tag{36}$$

Expression (36) and its obvious extension to general multitime position measurements serves as the basic formula for the subsequent analysis. Its essential features are

[11] $\mathscr{H}_1$ is norm dense in $\mathscr{H}$. Control of the limit $\lambda \to 0$ for $\psi \in \mathscr{H}_1$ extends to all $\psi \in \mathscr{H}$ by the uniform boundedness of the operators $n_\Delta(\mathscr{Q}(\tau))$.

[12] A factor $(2\pi\hbar)^{-\nu/2}$ has been absorbed into $\psi(x)$.

1. The integrand in $k_1,\ldots,k_N$ contains an imaginary quadratic exponential, where the quadratic form $\mathscr{K}$, given by (33), does not depend on the regions $\Delta$ or times $\tau$, and is in fact the same as the quadratic form studied in ref. 12, § 6 and § 8, with the same conclusion applying.

2. The remaining exponential is *linear* in $k_1,\ldots,k_N$ and will not play a role in the basic estimates controlling the $\lambda \to 0$ limit (although it *will* play a role in the actual limiting value).

3. The integrals over $\kappa_1,\ldots,\kappa_4$ will be bounded in terms of $\prod_{j=1}^4 \|\widehat{\eta_{\Delta_j}}\|_1$ and the integrals over $p, x$ by $\|\hat{\psi}\|_1\ \|\psi\|_1$.

**D.** The above analysis for a single field $v(x)$ will now be applied to a random field: For each $x$, $v(x)$ is a random variable.[13] We shall for the momentum suppose the random field satisfies

$$\left\langle \exp\left[\beta \int dp\ |\hat{v}(p)|\right]\right\rangle < \infty \qquad \text{for all real } \beta \tag{37}$$

The sum of the absolute value of the terms in (36) is bounded by

$$\|\hat{\psi}\|_1\ \|\psi\|_1\ \|\hat{\eta}_{\Delta_1}\|_1^2\ \|\hat{\eta}_{\Delta_2}\|_1^2 \sum_n (1/n!)\left[\lambda^{-2}\tau_1(2\lambda/h)\int dk\ |\hat{v}(k)|\right]^n$$

$$\times \sum_{n'} (1/n'!)\left[\lambda^{-2}(\tau_2-\tau_1)(2\lambda/h)\int dk\ |\hat{v}(k)|\right]^{n'}$$

$$= \|\hat{\psi}\|_1\ \|\psi\|_1\ \|\hat{\eta}_{\Delta_1}\|_1^2\ \|\hat{\eta}_{\Delta_2}\|_1^2 \exp\left[2\tau_2/(h\lambda)\int dk\ |\hat{v}(k)|\right]$$

and

$$\left\langle \exp\left[2\tau_2/(h\lambda)\int dk\ |\hat{v}(k)|\right]\right\rangle < \infty$$

by (37). Hence summation over $N$ and averaging over $v$ may be interchanged. Thus

$$\langle P_\psi(\Delta_1,\tau_1;\Delta_2,\tau_2)\rangle$$

$$= \sum_{N=0}^{\infty} (-i\lambda/h)^N \sum_{n+n'=N}\ \sum_{\phi\in\Phi_n}\ \sum_{\phi'\in\Phi_{n'}} (-1)^{r+r'}$$

$$\times \int dt_1\cdots dt_N\, d\kappa_1\cdots d\kappa_4\, \widehat{\eta_{\Delta_1}}(\kappa_1)\,\widehat{\eta_{\Delta_2}}(\kappa_2)\,\widehat{\eta_{\Delta_2}}(\kappa_3)\,\widehat{\eta_{\Delta_1}}(\kappa_4)$$

$$\times \int dp\, dx\, \exp[-ip\cdot x/h]\,\overline{\hat{\psi}(p)}\,\psi(x)$$

[13] Let $(\Omega, d\mu)$ be a probability space. The random field $v$ is a real-valued measurable function on $(R^{\nu}\times\Omega, dx\times d\mu)$.

$$\times \exp\left[i \sum_{a=1}^{4} (\lambda^2 x + (\tau_a/m)p) \cdot \kappa_a\right] \exp[-(i\hbar/2m)\mathscr{K}_1]$$

$$\times \int dk_1 \cdots dk_N \langle \hat{v}(k_1) \cdots \hat{v}(k_N)\rangle \exp[-(i\hbar/2m)\mathscr{K}]$$

$$\times \exp i\left[\sum_{j=1}^{N} (x + (t_j/m)p) \cdot k_j - (\hbar/2m)\mathscr{K}_2\right] \tag{38}$$

Note that the interchange of integration over $k_1, \ldots, k_N$ and average over $v$ is justified since

$$\int dk_1 \cdots dk_N \langle |\hat{v}(k_1)| \cdots |\hat{v}(k_N)|\rangle < \infty$$

Similarly, *the average of a product of any number of such $P_\psi$'s will also be given by averaging the product of all $\hat{v}$'s occuring in the integrands.*

**E.** The formula (38) for $\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)\rangle$ and its extensions will now be generalized to unbounded, translation-invariant random fields by the methods of ref. 12, § 4 and § 5. The conditions on the random field are as follows.

1. *Translation Invariance:*

$$\langle F_1(v(x_1)) \cdots F_k(v(x_k))\rangle = \langle F_1(v(x_1 + a)) \cdots F_k(v(x_k + a))\rangle$$

for any bounded measurable functions $F_1, \ldots, F_k$ on $R$ and any $a \in R^\nu$.

2. *Regularity:*

$$\langle e^{\beta v(0)}\rangle < \infty \qquad \text{for all real } \beta$$

3. *Zero Mean*[14]*:*

$$\langle v(0)\rangle = 0$$

4. *Cluster Property:*

$$\langle v(x_1) \cdots v(x_r)\, v(x_{r+1})\rangle^T = \gamma_r(y_1, \ldots, y_r)$$

where $y_i = x_i - x_{r+1}$ and the truncated correlation functions $\gamma_r$ and their Fourier transforms $\hat{\gamma}_r$ satisfy

$$\|\gamma_r\|_1 < \infty, \qquad \|\hat{\gamma}_r\|_1 < C^r \qquad \text{for some} \quad C > 0$$

[14] If $\langle v(0)\rangle \neq 0$, then this just adds a constant to the Hamiltonian and does not affect the later results.

From condition 4 we have

$$\langle \hat{v}(k_1) \cdots \hat{v}(k_r)\, \hat{v}(k_{r+1}) \rangle^T = \hat{\gamma}_r(k_1, \dots, k_r)\, \delta(k_1 + \cdots k_r + k_{r+1}) \tag{39}$$

Let $\mathscr{P}_N$ denote the set of all partitions $P$ of $\{1, \dots, N\}$. A partition $P$ which partitions $\{1, \dots, N\}$ into $M$ subsets is represented as $(s_M = N - M)$

$$P = \{\beta(1), \dots, \beta(s_1), \mu(1)\}, \dots, \{\beta(s_{M-1}+1), \dots, \beta(s_M), \mu(M)\}$$

where

$$\beta(s_{j-1}+1) < \cdots < \beta(s_j) < \mu(j)$$

Correspondingly, we decompose the correlation function $\langle \hat{v}(k_1) \cdots \hat{v}(k_N) \rangle$ into a sum of products of truncated functions

$$\begin{aligned}
&\langle \hat{v}(k_1) \cdots \hat{v}(k_N) \rangle \\
&\quad = \sum_{P \in \mathscr{P}_N} \langle \hat{v}(k_{\beta(1)}) \cdots \hat{v}(k_{\beta(s_1)})\, \hat{v}(k_{\mu(1)}) \rangle^T \cdots \\
&\qquad \times \langle \hat{v}(k_{\beta(s_{M-1}+1)}) \cdots \hat{v}(k_{\beta(s_M)})\, \hat{v}(k_{\mu(M)}) \rangle^T \\
&\quad = \sum_{P \in \mathscr{P}_N} \hat{\gamma}_{r_1}(k_{\beta(1)}, \dots, k_{\beta(s_1)}) \cdots \hat{\gamma}_{r_M}(k_{\beta(s_{M-1}+1)}, \dots, k_{\beta(s_M)}) \\
&\qquad \times \delta(k_{\beta(1)} + \cdots + k_{\beta(s_1)} + k_{\mu(1)}) \cdots \\
&\qquad \times \delta(k_{\beta(s_{M-1}+1)} + \cdots + k_{\beta(s_M)} + k_{\mu(M)})
\end{aligned} \tag{40}$$

where $r_j = s_j - s_{j-1}$ $(s_0 = 0)$.

The analysis of ref. 12, § 4 and § 5, shows that the formula (38) holds also for the random field $v(x)$, with $\langle \hat{v}(k_1) \cdots \hat{v}(k_N) \rangle$ given by (40). We thus obtain

$$\begin{aligned}
&\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2) \rangle \\
&\quad = \sum_{N=0}^{\infty} (-i\lambda/h)^N \sum_{n+n'=N} \sum_{\phi \in \Phi_n} \sum_{\phi' \in \Phi_{n'}} (-1)^{r+r'} \\
&\qquad \times \sum_{P \in \mathscr{P}_N} \int dt_1 \cdots dt_N\, d\kappa_1 \cdots d\kappa_4\, \widehat{\eta_{\Delta_1}}(\kappa_1)\, \widehat{\eta_{\Delta_2}}(\kappa_2) \\
&\qquad \times \widehat{\eta_{\Delta_2}}(\kappa_3)\, \widehat{\eta_{\Delta_1}}(\kappa_4) \int dp\, dx\, \exp[-ip \cdot x/h]\, \overline{\hat{\psi}(p)}\, \psi(x) \\
&\qquad \times \exp\left[ i \sum_{a=1}^{4} (\lambda^2 x + (\tau_a/m) p) \cdot \kappa_a \right] \exp[-(ih/2m) \mathscr{K}_1] \\
&\qquad \times \int dk_1 \cdots dk_N\, \delta(k_{\beta(1)} + \cdots + k_{\beta(s_1)} + k_{\mu(1)}) \cdots
\end{aligned}$$

$$\times \delta(k_{\beta(s_{M-1}+1)} + \cdots + k_{\beta(s_M)} + k_{\mu(M)})$$
$$\times \hat{\gamma}_{r_1}(k_{\beta(1)},..., k_{\beta(s_1)}) \cdots \hat{\gamma}_{r_M}(k_{\beta(s_{M-1}+1)},..., k_{\beta(s_M)})$$
$$\times \exp[-(i\hbar/2m)\mathscr{K}] \exp i\left[\sum_{j=1}^{N} (x + (t_j/m)p) \cdot k_j - (\hbar/2m)\mathscr{K}_2\right] \tag{41}$$

The variables $k_{\mu(1)},..., k_{\mu(M)}$ are eliminated by means of the $\delta$-functions.

*The obvious extension of this formula holds for* $\langle P_\psi(\Delta_1, \tau_1;...; \Delta_a, \tau_a)\rangle$ *and for the average of products of* $P_\psi$*'s.*

**Remark.** The derivation of (41) proceeds by introducing (ref. 12, Definition 4.4) a cutoff random field

$$v_\delta(x) = \delta^{-\nu} \int dy\, h(\delta^{-1}(x-y))\, e^{-\delta y^2/2} v(y)$$

where $h$ is a smooth nonnegative function of compact support with $\int dx\, h(x) = 1$. Then

$$\widehat{v_\delta}(k) = \hat{h}(\delta k) \int dy\, e^{-ip \cdot y} e^{-\delta y^2/2} v(y)$$

and

$$\int \delta k\, |\widehat{v_\delta}(k)| \leqslant \int dk\, |\hat{h}(\delta k)| \int dy\, e^{-\delta y^2/2}\, |v(y)|$$
$$= C(\delta/(2\pi))^{\nu/2} \int dy\, e^{-dy^2/2}\, |v(y)|$$

Thus by Jensen's inequality

$$\left\langle \exp\left[\beta \int dk\, |\widehat{v_\delta}(k)|\right]\right\rangle \leqslant (\delta/(2\pi))^{\nu/2} \int dy \exp(-\delta y^2/2) \langle \exp[\beta C\, |v(y)|]\rangle$$
$$= \langle \exp[\beta C\, |v(0)|]\rangle$$

which is finite for all $\beta$ by condition 2. Thus the condition (37) of Section 3.2D holds and consequently (38) holds for $v_\delta$. The limit $\delta \to 0$ for the integrals in (38) is controlled by Lemma 5.4 and Proposition 5.5 of ref. 12. The limit $\delta \to 0$ for $\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)\rangle$ is controlled by showing (ref. 12, Proposition 4.7) $v_\delta(x)\,\psi(x) \to v(x)\,\psi(x)$ in $\mathscr{H}$ for $\psi \in C_0^\infty$, with probability 1, which implies (ref. 12, Theorem 4.6) strong convergence of the unitary time evolutions with probability 1.

**Remark.** In ref. 12, § 4.1, the essential self-adjointness of $H_0 + \lambda v(q)$ on $C_0^\infty$ was derived under the conditions $\langle |v(0)|^k \rangle < \infty$ for some $k > 2$ and $\langle \exp[-\delta v(0)] \rangle < \infty$ for some $\delta > 0$. In ref. 3, Corollary V.3.3, essential self-adjointness is derived under the condition $\langle |v(0)|^k \rangle < \infty$ for some $k > r(\nu)$, where $\nu$ is the number of space dimensions and

$$r(\nu) = \begin{cases} 2 + \nu/2 & \text{if} \quad \nu \leqslant 3 \\ 4 & \text{if} \quad \nu = 4 \\ \nu & \text{if} \quad \nu \geqslant 5 \end{cases}$$

### 3.3. The Large-Space-Time, Weak-Coupling Limit

The individual terms of the perturbative expansion (41) of $\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2) \rangle$ can be controlled in the limit $\lambda \to 0$ provided the random field $v(x)$ satisfies, in addition to the conditions in Section 3.2E, a generalized cluster property formulated in terms of the partial Fourier transforms of the truncated functions $\gamma_r(y_1,..., y_r)$. A partial Fourier transform $\tilde{\gamma}_r$ is a transform with respect to a *subset* of variables $y_{\alpha_1},..., y_{\alpha_l}$:

$$\int dy_{\alpha_1} \cdots dy_{\alpha_l} \exp[-i(k_{\alpha_1} \cdot y_{\alpha_1} + \cdots + k_{\alpha_l} \cdot y_{\alpha_l})] \, \gamma_r(y_1,..., y_r)$$

Let

$$\|\gamma_r\| = \sup \|\tilde{\gamma}_r\|_1$$

where the supremum is over all partial Fourier transforms (ref. 12, Remark 1.9).

5. *Generalized Cluster Property:* The truncated correlation functions $\gamma_r$ satisfy

$$\|\gamma_r\| < \infty$$

In order to control the *sum* of the perturbative terms, the random field must be Gaussian.

6. *Gaussian Property:* Truncated correlation functions of order greater than second order are zero:

$$\gamma_r = 0 \qquad \text{if} \quad r > 1$$

**A.** An individual term in the perturbative expansion (41) has the form

$$\lambda^N \int dt_1 \cdots dt_N \, \mathscr{F}_\lambda(t_1,..., t_N) \tag{42}$$

where

$$\begin{aligned}
&\mathscr{F}_\lambda(t_1,\ldots,t_N)\\
&\quad=(-i/\hbar)^N(-1)^{r+r'}\int d\kappa_1\cdots d\kappa_4\prod_{a=1}^{4}\widehat{\eta_{\Delta_a}}(\kappa_a)\\
&\quad\times\int dp\,dx\exp[-ip\cdot x/\hbar]\,\overline{\hat{\psi}(p)}\,\psi(x)\\
&\quad\times\exp\left[i\sum_{a=1}^{4}(\lambda^2x+(\tau_a/m)p)\cdot\kappa_a\right]\\
&\quad\times\exp\left[-i\hbar/(2m)\,\lambda^2\sum_{a,b=1}^{4}\tau_{a\vee b}\kappa_a\kappa_b\right]\int dk_{\beta(1)}\cdots dk_{\beta(N-M)}\\
&\quad\times\hat{\gamma}_{r_1}(k_{\beta(1)},\ldots,k_{\beta(s_1)})\cdots\hat{\gamma}_{r_M}(k_{\beta(s_{M-1}+1)},\ldots,k_{\beta(s_M)})\\
&\quad\times\exp[-i\hbar/(2m)\mathscr{M}]\exp[i\mathscr{A}_\lambda]
\end{aligned}\tag{43}$$

The quadratic form $\mathscr{M}=\Sigma_{l,l'=1}^{N-M}\mathscr{M}_{ll'}k_{\beta(l)}\cdot k_{\beta(l')}$, where the associated matrix (also denoted $\mathscr{M}$) is

$$\mathscr{M}_{ll'}=t_{\beta(l)\vee\beta(l')}+t_{\alpha(l)\vee\alpha(l')}-t_{\alpha(l')\vee\beta(l)}-t_{\alpha(l)\vee\beta(l')}\tag{44}$$

and, if $s_{j-1}<l<s_j$, we have defined $\alpha(l)=\mu(j)$.

The exponent $\mathscr{A}_\lambda$ is linear in $k_{\beta(1)},\ldots,k_{\beta(N-M)}$ and is given by

$$\begin{aligned}
\mathscr{A}_\lambda=(p/m)\cdot\sum_{l=1}^{N-M}[t_{\beta(l)}-t_{\alpha(l)}]\,k_{\beta(l)}-(\hbar/m)\sum_{a=1}^{4}\kappa_a\cdot\Bigg[\tau_a\sum_{\beta(l)\leqslant j(a)<\alpha(l)}k_{\beta(l)}\\
+\lambda^2\sum_{j(a)<\beta(l)}[t_{\beta(l)}-t_{\alpha(l)}]\,k_{\beta(l)}-\lambda^2\sum_{\beta(l)\leqslant j(a)<\alpha(l)}t_{\alpha(l)}k_{\beta(l)}\Bigg]
\end{aligned}\tag{45}$$

An upper bound for $\mathscr{F}_\lambda$ is given by

$$|\mathscr{F}_\lambda(t_1,\ldots,t_N)|\leqslant\hbar^{-N}\,\|\widehat{\eta_{\Delta_1}}\|_1^2\,\|\widehat{\eta_{\Delta_2}}\|_1^2\,\|\hat{\psi}\|_1\,\|\psi\|_1\,\|\gamma_{r_1}\|\cdots\|\gamma_{r_M}\|\,[\max\mathscr{M}]^{-\nu/2}\tag{46}$$

where

$$\max\mathscr{M}=\sup_{\mathscr{M}'\prec\mathscr{M}}|\det\mathscr{M}'|$$

the supremum of determinants of *submatrices* of $\mathscr{M}$ (ref. 12, § 1; ref. 15, § 4).

Because the matrix $\mathscr{M}$ is the same as the matrix considered in ref. 12, § 5 and § 6, we may take over the results obtained there. (See also refs. 15 and 19.)

*Any term in the perturbative expansion (41) which contains a truncated function $\gamma_r$ of order higher than second order tends to zero as $\lambda \to 0$.*

We shall therefore consider those terms in (41) containing only two-point truncated functions of the random field. The partition $P$ associated with such a terms has the form

$$P = \{\beta(1), \alpha(1)\},..., \{\beta(N/2), \alpha(N/2)\} \tag{47}$$

The partition thus defines a pairing of $\{1,..., N\}$. We say the pair $\{\beta(j), \alpha(j)\}$ is *consecutive* if $t_{\beta(j)}, t_{\alpha(j)}$ are consecutive. In other words, referring to the variables $u_1,..., u_n$; $s_1,..., s_{n'}$ in (25), $u_j$ is consecutive with $u_{j+1}$, and $s_j$ is consecutive with $s_{j+1}$. However, $u_1$ is not consecutive with $s_{n'}$, since $\lambda^{-2}\tau_1$ lies between them. A term in (41) will be called nonconsecutive if it contains at least one nonconsecutive pair; otherwise it will be called *consecutive*. Then (ref. 12, § 6.2; ref. 15):

*Any nonconsecutive term in (41) tends to zero as $\lambda \to 0$.*

**B.** We shall now consider a consecutive term in (41). It is convenient to introduce the totally ordered time variables $0 \leqslant T_N \leqslant T_{N-1} \leqslant \cdots \leqslant T_1$. In terms of the variables $u_1,..., u_n$; $s_1,..., s_{n'}$ in (25)

$$T_1 = s_1, \quad T_2 = s_2,..., \quad T_{n'} = s_{n'}, \quad T_{n'+1} = u_1,..., \quad T_N = u_n$$

Then $0 \leqslant T_N \leqslant \cdots \leqslant T_{n'+1} \leqslant \lambda^{-2}\tau_1 \leqslant T_{n'} \leqslant \cdots \leqslant T_1 \leqslant \lambda^{-2}\tau_2$ and notice that for a consecutive term both $n$ and $n'$ must be even. As the term is consecutive, $T_{2l-1}$ is paired with $T_{2l}$, $l = 1,..., N/2$. Enumerate the pairs so that $\{t_{\beta(l)}, t_{\alpha(l)}\} = \{T_{2l-1}, T_{2l}\}$. Set

$$\varepsilon_l = \begin{cases} 1 & \text{if} \quad t_{\beta(l)} = T_{2l-1} \\ -1 & \text{if} \quad t_{\beta(l)} = T_{2l} \end{cases}$$

Then

$$t_{\beta(l)} - t_{\alpha(l)} = \varepsilon_l(T_{2l-1} - T_{2l}) \tag{48}$$

$$t_{\alpha(l)} = T_{2l-1} - (1 + \varepsilon_l)/2(T_{2l-1} - T_{2l}) \tag{49}$$

Introduce the variables $w_l = \lambda^2 T_{2l-1}$, $v_l = T_{2l-1} - T_{2l}$, $l = 1,..., N/2$, and express $\mathscr{M}$ of (44) and $\mathscr{A}_\lambda$ of (45) in terms of the variables $v$, $w$. To do so, define

$$\sigma_l = \begin{cases} 1 & \text{if} \quad \exists a \text{ such that } \beta(l) \leqslant j(a) < \alpha(l) \\ 0 & \text{otherwise} \end{cases}$$

(See Fig. 1. This definition is equivalent to that in ref. 12, § 6.3.3, and ref. 15, § 6.) Note that then $(-1)^{r+r'} = (-1)^{\Sigma \sigma_l}$. Then, using (48) and expression (44), we have

$$\mathscr{M}_{ll} = t_{\beta(l)} - t_{\alpha(l)} = \varepsilon_l v_l$$

$$\mathscr{M}_{ll'} = \sigma(l')\, \varepsilon(l)\, v_l \qquad \text{for} \quad l' > l$$

Hence

$$\mathscr{M} = \sum_{l=1}^{N/2} \varepsilon_l v_l k_{\beta(l)} \cdot \left\{ k_{\beta(l)} + 2 \sum_{l'>l} \sigma_{l'} k_{\beta(l')} \right\} \tag{50}$$

In terms of the variables $v$, $w$, the linear form $\mathscr{A}_\lambda$ is given by

$$\mathscr{A}_\lambda = (p/m) \cdot \sum_{l=1}^{N/2} \varepsilon_l v_l k_{\beta(l)} - (\hbar/m) \sum_{a=1}^{4} \kappa_a \cdot \sum_{\beta(l) \leqslant j(a) < \alpha(l)} (\tau_a - w_l)\, k_{\beta(l)}$$
$$- \lambda^2 (\hbar/m) \sum_{a=1}^{4} \kappa_a \cdot \left\{ \sum_{j(a) < \beta(l)} \varepsilon_l v_l k_{\beta(l)} + \sum_{\beta(l) \leqslant j(a) < \alpha(l)} [(1+\varepsilon_l)/2]\, v_l k_{\beta(l)} \right\}$$

which converges to $\mathscr{A}_0$ as $\lambda \to 0$, where

$$\mathscr{A}_0 = (p/m) \cdot \sum_{l=1}^{N/2} \varepsilon_l v_l k_{\beta(l)} - (\hbar/m) \sum_{a=1}^{4} \kappa_a \cdot \sum_{\beta(l) \leqslant j(a) < \alpha(l)} (\tau_a - w_l)\, k_{\beta(l)}$$
$$= (p/m) \cdot \sum_{l=1}^{N/2} \varepsilon_l v_l k_{\beta(l)} - (\kappa_2 + \kappa_3) \cdot (\hbar/m) \sum_{l} (\tau_2 - w_l)\, \sigma_l k_{\beta(l)}$$
$$- (\kappa_1 + \kappa_4) \cdot (\hbar/m) \sum_{l > n'/2} (\tau_1 - w_l)\, \sigma_l k_{\beta(l)} \tag{51}$$

Using the variables $v$, $w$, the expression (42) becomes

$$\int_{\tau_1}^{\tau_2} dw_1 \int_{\tau_1}^{w_1} dw_2 \cdots \int_{\tau_1}^{w_{n'/2-1}} dw_{n'/2} \int_0^{\tau_1} dw_{n'/2+1} \cdots$$
$$\times \int_0^{w_{N/2-1}} dw_{N/2} \int_0^{\lambda^{-2}(w_1 - w_2)} dv_1 \cdots$$
$$\times \int_0^{\lambda^{-2}(w_{n'/2} - \tau_1)} dv_{n'/2} \int_0^{\lambda^{-2}(w_{n'/2+1} - w_{n'/2+2})} \cdots \int_0^{\lambda^{-2} w_{N/2}} dv_{N/2}\, \mathscr{F}_\lambda(w, v) \tag{52}$$

Since the upper bound (46) is integrable over the range $0 \leqslant v < \infty$ (ref. 12, § 1 and § 6.2), dominated convergence yields the limit

$$\int_{\tau_1}^{\tau_2} dw_1 \cdots \int_{\tau_1}^{w_{n'/2-1}} dw_{n'/2} \int_0^{\tau_1} dw_{n'/2+1} \cdots$$
$$\times \int_0^{w_{N/2-1}} dw_{N/2} \int_0^{\infty} dv_1 \cdots \int_0^{\infty} dv_{N/2}\, \mathscr{F}_0(w, v) \tag{53}$$

where

$$\mathscr{F}_0(w, v) = (-1/\hbar^2)^{N/2} (-1)^{\Sigma \sigma_l} \int dp \, |\hat{\psi}(p)|^2$$
$$\times \int dk_{\beta(1)} \cdots dk_{\beta(N/2)} \hat{\gamma}_1(k_{\beta(1)}) \cdots \hat{\gamma}_1(k_{\beta(N/2)})$$
$$\times \eta^2_{\Delta_1} \left( (\tau_1/m) p - (\hbar/m) \sum_{l > n'/2} (\tau_1 - w_l) \, \sigma_l k_{\beta(l)} \right)$$
$$\times \eta^2_{\Delta_2} \left( (\tau_2/m) p - (\hbar/m) \sum_l (\tau_2 - w_l) \, \sigma_l k_{\beta(l)} \right)$$
$$\times \exp \left[ i(\hbar/2m) \sum_{l=1}^{N/2} \varepsilon_l v_l k_{\beta(l)} \cdot \left\{ 2p/\hbar - k_{\beta(l)} - 2 \sum_{l' > l} \sigma_{l'} k_{\beta(l')} \right\} \right] \tag{54}$$

Summing over the $\varepsilon$'s is equivalent to extending the $v$-integration range to $(-\infty, \infty)$, and using[15]

$$\int_{-\infty}^{\infty} ds \exp[i(\hbar/2m) sy)] = (4\pi m/\hbar) \, \delta(y)$$

the expression (53) becomes

$$(-4\pi m/\hbar^3)^N (-1)^{\Sigma \sigma_l} \int_{\tau_1}^{\tau_2} dw_1 \cdots \int_0^{\tau_1} dw_{n'+1} \cdots$$
$$\times \int_0^{w_{N-1}} dw_N \int dp \, |\hat{\psi}(p)|^2 \int dk_1 \cdots dk_N \, \gamma(k_1) \cdots$$
$$\times \gamma(k_N) \eta^2_{\Delta_1} \left( (\tau_1/m) p - (\hbar/m) \sum_{l > n'} (\tau_1 - w_l) \, \sigma_l k_{\beta(l)} \right)$$
$$\times \eta^2_{\Delta_2} \left( (\tau_2/m) p - (\hbar/m) \sum_l (\tau_2 - w_l) \, \sigma_l k_{\beta(l)} \right)$$
$$\times \prod_{l=1}^{N} \delta \left( k_l^2 - 2(p/\hbar) \cdot k_l + 2 \sum_{l' > l} \sigma_{l'} k_l \cdot k_{l'} \right) \tag{55}$$

*We have now denoted* $k_{\beta(l)}$ *by* $k_l$, $\hat{\gamma}_1$ *by* $\gamma$, $N/2$ *by* $N$, $n'/2$ *by* $n'$, *and* $n/2$ *by* $n$.[16]

[15] This is rigorously justified as in ref. 12, § 6.3.4.

[16] Although $n$ is not explicitly expressed in (55), recall that $N = n + n'$.

## 3.4. Summing the Perturbation Series

In Section 3.3 the weak-coupling limit was taken term by term in perturbation theory. The sum of the terms can be controlled if the random field is Gaussian, which we henceforth assume. An individual term in the perturbative expansion has the form (42), where $\mathscr{F}_\lambda(t_1,..., t_N)$ has the upper bound (46). The form of the bound, involving max $\mathscr{M}$, is the same as that considered in ref. 12, § 6.3.2, leading to the same conclusion. For given values $n_1$ and $n_2$ in (41), there are $2^{n_1}2^{n_2}$ configurations, and for each there are $(n_1+n_2-1)!!$ pairings. With reference to (25) and (46), the upper bound of each term is

$$\lambda^N\hbar^{-N}\,\|\widehat{\eta_{\Delta_1}}\|_1^2\,\|\,\|\widehat{\eta_{\Delta_2}}\|_1^2\,\|\hat{\psi}\|_1\,\|\psi\|_1\,\|\gamma\|^{N/2}$$
$$\times\int_0^{\lambda^{-2}\tau_1}du_1\cdots\int_0^{u_{n-1}}du_n\int_{\lambda^{-2}\tau_1}^{\lambda^{-2}\tau_2}ds_1\cdots\int_{\lambda^{-2}\tau_1}^{s_{n'-1}}ds_{n'}\,[\max\mathscr{M}]^{-\nu/2} \tag{56}$$

where $N=n_1+n_2$. The bound (56) is increased if we replace the integral by

$$\int_0^{\lambda^{-2}\tau_2}ds_1\cdots\int_0^{s_{n'-1}}ds_{n'}\int_0^{s_{n'}}du_1\cdots\int_0^{u_{n-1}}du_n[\max\mathscr{M}]^{-\nu/2} \tag{57}$$

The properties of $\mathscr{M}$ lead to a bound for (57) given by (ref. 12, § 1.3)

$$[C_{\nu/2}^{N/2}](\lambda^{-2}\tau_2)^{N/2}/(N/2)!$$

where $C_{\nu/2}=2\nu(\nu-2)^{-1}$. Hence (56) is bounded by

$$C[\tau_2\,\|\gamma\|\,C_{\nu/2}/\hbar^2]^{N/2}/(N/2)! \tag{58}$$

where

$$C=\|\widehat{\eta_{\Delta_1}}\|_1^2\,\|\widehat{\eta_{\Delta_2}}\|_1^2\,\|\hat{\psi}\|_1\,\|\psi\|_1$$

As there are $N+1$ choices for $n_1, n_2$ with $n_1+n_2=N$, the terms of order $N$ are bounded by

$$2^N(N-1)!!\,(N+1)\,C\,[\tau_2\,\|\gamma\|\,C_{\nu/2}/\hbar^2]^{N/2}/(N/2)! \tag{59}$$

Using $(N-1)!!<2^{N/2}(N/2)!$, we find that (59) is bounded by $C(N+1)[\tau_2 8\,\|\gamma\|\,C_{\nu/2}/\hbar^2]^{N/2}$. Hence the sum over $N$ of (59) converges for $\tau_2<\tau_0$, where

$$\tau_0^{-1}=8\,C_{\nu/2}\,\|\gamma\|/\hbar^2 \tag{60}$$

We recall that $\|\gamma\|=\max\{\|\gamma\|_1, \|\hat{\gamma}\|_1\}$.

*The same methods yield convergence of the perturbation series for* $\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2; ...; \Delta_a, \tau_a)\rangle$, *where* $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_a < \tau_0$.

We extend this convergence, proved for $\psi, \hat{\psi} \in \mathscr{H}_1$ (Section 3.2C), to all $\psi \in \mathscr{H}$ by the uniform boundedness of the operators $\eta_{\Delta_j}(\mathscr{Q}(\tau_j))$.

### 3.5. Large-Scale, Weak-Coupling Limit with Probability 1

**A.** In Section 3.4 we showed the convergence of $\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)\rangle$ to the sum over $n$ and $n'$ of (55) for $0 \leqslant \tau_1 \leqslant \tau_2 < \tau_0$. The techniques of ref. 12, § 8, can be used to extend this to the convergence of $P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)$ to $\lim_{\lambda \to 0}\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)\rangle$ as $L^p$ functions of the random field, for $1 \leqslant p < \infty$. This is done by using $|P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)| \leqslant 1$ together with the $L^2$-convergence of $P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)$ to $\lim_{\lambda \to 0}\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)\rangle$, which follows from

$$\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)^2\rangle - \langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)\rangle^2 \to 0 \tag{61}$$

**B.** The proof of (61) is shown term by term in perturbation theory, by showing that *any term in the perturbative expansion of* $\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)^2\rangle$ *which does not factorize tends to zero as* $\lambda \to 0$. The proof of this follows by the methods of ref. 12, § 1 and § 8, which depend on properties of the matrix $\mathscr{M}$. The proof of (61) is now completed by showing convergence of the perturbative expansion for $\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)^2\rangle$. Indeed, as in ref. 12, § 8.4.6, if the order of the perturbative terms for the two factors of $P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)$ is $n_1$, $n_2$ and $n'_1$, $n'_2$, with $N = n_1 + n_2 + n'_1 + n'_2$, then the number of configurations is $2^N$, the number of pairings is $(N-1)!!$, and the time integral of $[\max \mathscr{M}]^{\nu/2}$ is bounded by

$$[(\lambda^{-2}\tau_2)^a/a!][(\lambda^{-2}\tau_2)^b/b!]\; C_{\nu/2}^{N/2} \tag{62}$$

for some $a, b$ with $a + b = N/2$. Using $[C^a/a!][C^b/b!] \leqslant (2C)^{a+b}/(a+b)!$, we obtain that (62) is bounded by

$$(2\lambda^{-2}\tau_2 C_{\nu/2})^{N/2}/(N/2)!$$

Thus, since

$$\sum_{n_1, n_2, n'_1, n'_2} (8C_{\nu/2}\, \|\gamma\|\, \tau_2/h^2)^{(n_1+n_2+n'_1+n'_2)/2} < \infty$$

for $\tau_2 < \tau_0$, the convergence of the series for $\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)^2\rangle$ is shown. In the same manner we conclude:

$P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2; ...; \Delta_a, \tau_a)$ *convergence to* $\lim_{\lambda \to 0}\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2; ...; \Delta_a, \tau_a)\rangle$ *as* $L^p$ *functions of the random field, for* $1 \leqslant p < \infty$.

**C.** It is generally true that if $f_\lambda$ converges to $f$ as $L^2$ functions, then there is a subsequence $\lambda_k \to 0$ such that $f_\lambda$ converges to $f$ *with probability* 1.[17] Consequently, there is a subset[18] of random fields $\Omega_0 \subset \Omega$ of measure 1 and a sequence $\lambda_k \to 0$, such that $P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)$ converges for fields in $\Omega_0$ to $\lim_{\lambda \to 0}\langle P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)\rangle$. This extends to a *countable* set of $\Delta$'s and $\tau$'s with all $\tau_j < \tau_0$. This also extends to a countable set of $\psi$'s, and since $\mathscr{H}$ is separable, to *all* $\psi \in \mathscr{H}$.

### 3.6. The Classical Jump Process

For a Gaussian random field and $\tau_2 < \tau_0$, the limit as $\lambda_k \to 0$ of $P_\psi(\Delta_1, \tau_1; \Delta_2, \tau_2)$ is, with probability 1, given by the sum over $n, n'$ of (55). This joint probability is in fact given by classical particle trajectories, where the particle moves with constant velocity between random jumps in direction (the speed remaining constant).

The trajectory in phase space of a freely moving classical particle is given by the straight lines $(x(\tau), p(\tau)) = (x + (\tau/m)p, p)$. Given an initial probability distribution $d\mu(x, p)$ for the particle and a function $f(x, p)$ on phase space, the expectation value (average value) of $f(x(\tau), p(\tau))$ at time $\tau$ is given by $\int d\mu(x, p)\, f(x + (\tau/m)p, p)$. The function $f(x + (\tau/m)p, p) = (T^0_\tau f)(x, p)$ is the *conditional expectation* of $f(x(\tau), p(\tau))$ given the initial values $(x(0) = x, p(0) = p)$. Of course in this case the motion is deterministic and $T^0_\tau$ defines a one-parameter evolution group.

For a Markov process with stationary transition probabilities, the conditional expectation determines a semigroup $T_\tau$ such that the conditional expectation of $f(x(\tau), p(\tau))$, given the initial values $(x(0) = x, p(0) = p)$, is $(T_\tau f)(x, p)$. Given an initial probability distribution $d\mu(x, p)$, the expectation value of $f(x(\tau), p(\tau))$ is

$$\langle f(x(\tau), p(\tau))\rangle = \int d\mu(x, p)(T_\tau f)(x, p) \tag{63}$$

Furthermore, by the Markov property, the expectation value of

$$f_1(x(\tau_1), p(\tau_1))\, f_2(x(\tau_2), p(\tau_2)) \qquad \text{with} \quad 0 \leqslant \tau_1 \leqslant \tau_2$$

is given by

$$\int d\mu(x, p)\; T_{\tau_1}\{f_1(T_{\tau_2 - \tau_1} f_2)\}(x, p) \tag{64}$$

[17] It is only necessary to choose $\lambda_k$ so that $\sum_k \| f_{\lambda_k} - f \|_2^2 < \infty$. Then $\sum_k | f_{\lambda_k} - f |^2$ is finite with probability 1, which implies $f_{\lambda_k} \to f$ with probability 1.

[18] That is, fields $v(x, \omega)$ with $\omega \in \Omega_0$.

Write $T_\tau = \exp[-\tau \mathscr{L}]$, where $\mathscr{L}$ is the generator of the semigroup. Suppose $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1$, where $\mathscr{L}_0$ is the generator of the free evolution $T_\tau^0$, and suppose $\mathscr{L}_1$ generates a pure jump process. That is, the particle remains fixed at a point $(x, p)$ of phase space until it jumps randomly to another point $(x', p')$ with some jumping rate. The process generated by $\mathscr{L}$ is then described as a particle moving with constant velocity between random jumps. As in Section 3.1, a perturbation expansion for $T_\tau$ may be developed:

$$\begin{aligned}
T_\tau &= \sum_{N=0}^{\infty} (-1)^N \int_0^\tau dw_1 \int_0^{w_1} dw_2 \cdots \int_0^{w_{N-1}} dw_N \\
&\quad \times T^0_{w_N} \mathscr{L}_1 T^0_{w_{N-1}-w_N} \mathscr{L}_1 \cdots T^0_{w_1 - w_2} \mathscr{L}_1 T^0_{\tau - w_1} \\
&= \sum_{N=0}^{\infty} (-1)^N \int_0^\tau dw_1 \int_0^{w_1} dw_2 \cdots \int_0^{w_{N-1}} dw_N \\
&\quad \times \mathscr{L}_1(w_N)\, \mathscr{L}_1(w_{N-1}) \cdots \mathscr{L}_1(w_1)\, T^0_\tau
\end{aligned} \tag{65}$$

where $\mathscr{L}_1(w) = T^0_w \mathscr{L}_1 T^0_{-w}$. {With $\mathscr{L}_1$ given by (66), $T_r$ may be *defined* by the perturbative expansion on all continuous functions on phase space. The series converges for all values of $\tau$, pointwise and uniformly on compact subsets of phase space, since the series for $(T_\tau f)(x, p)$ depends only on the values of $f(x', p')$ for $|p'| = |p|$ and $|x' - x| \leqslant \tau\, |p|$. Furthermore, $|(\mathscr{L}_1 f)(x, p)| \leqslant C(|p|)\, \sup_{|p'|=|p|} |f(x, p')|$.}

Let $\mathscr{L}_1$ generate a pure jump process in momentum space (the coordinate $x$ being unchanged), defined by

$$\begin{aligned}
(\mathscr{L}_1 f)(x, p) = \sum_{\sigma = 0, 1} (-1)^\sigma (4\pi m/\hbar^3) \\
\times \int dk\, \gamma(k)\, f(x, p - \sigma \hbar k)\, \delta(k^2 - (2/\hbar) p \cdot k)
\end{aligned} \tag{66}$$

(See ref. 12, § 6.3.4, ref. 15, § 6; and refs. 6 and 19.) Then $\hbar k$ represents the transferred momentum and the $\delta$-function ensures conservation of energy. Then

$$\begin{aligned}
(\mathscr{L}_1(w) f)(x, p) = \sum_\sigma (-1)^\sigma (4\pi m/\hbar^3) \\
\times \int dk\, \gamma(k)\, f(x + (w/m)\, \sigma \hbar k,\, p - \sigma \hbar k)\, \delta(k^2 - (2/\hbar)\, p \cdot k)
\end{aligned}$$

Similarly

$$[\mathscr{L}_1(w_2)\,\mathscr{L}_1(w_1)f](x,p)$$

$$= \sum_{\sigma_1,\sigma_2} (-1)^{\sigma_1+\sigma_2} (4\pi m/\hbar^3)^2 \int dk_1\, dk_2\, \gamma(k_1)\, \gamma(k_2)$$

$$\times f(x + (w_2/m)\sigma_2 \hbar k_2 + (w_1/m)\,\sigma_1 \hbar k_1,\, p - \sigma_2 \hbar k_2 - \sigma_1 \hbar k_1)$$

$$\times \delta(k_2^2 - (2/\hbar) p\cdot k_2)\, \delta(k_1^2 - (2/\hbar) p\cdot k_1 + 2\sigma_2 k_2 \cdot k_1)$$

It is also straightforward to show that

$$(-1)^N [\mathscr{L}_1(\omega_N)\cdots \mathscr{L}_1(w_1)\, T_\tau^0 f](x,p) \tag{67}$$

[note the presence of $T_\tau^0$ in (67)] is equal to

$$\sum_{\sigma_1,\ldots,\sigma_N} (-4\pi m/\hbar^3)^N (-1)^{\Sigma \sigma_l} \int dk_1 \cdots dk_N\, \gamma(k_1)\cdots\gamma(k_N)$$

$$\times f\left(x + (\tau/m)p - (\hbar/m) \sum_{j=1}^{N} (\tau - w_j)\,\sigma_j k_j,\, p - \hbar \sum_{j=1}^{N} \sigma_j k_j\right)$$

$$\times \prod_{l=1}^{N} \delta\left(k_l^2 - 2(p/\hbar)\cdot k_l + 2\sum_{l'>l} \sigma_{l'} k_l \cdot k_{l'}\right) \tag{68}$$

By comparing (68) with (55), it is seen that

$$\lim_{\lambda\to 0} P_\psi(\Delta_1,\tau_1;\Delta_2,\tau_2) = \int dp\, |\psi(p)|^2\, T_{\tau_1}\{\eta_{\Delta_1}^2(T_{\tau_2-\tau_1}\eta_{\Delta_2}^2)\}(0,p) \tag{69}$$

which is the expectation value $\langle \eta_{\Delta_1}(x(\tau_1))^2\, \eta_{\Delta_2}(x(\tau_2))^2\rangle$ of the jump process with initial distribution $\delta(x)\,dx\,|\psi(p)|^2\,dp$. With the interpretation of $\eta_\Delta^2(x)$ discussed at the end of Section 1.2, we see that the joint distribution of multiple-time position measurements is, in the limit $\lambda\to 0$, reproduced by the classical jump process.

The convergence with probability 1, (69), may be expressed in the form

$$\lim_{\lambda\to 0} (\psi, \eta_{\Delta_1}(\mathscr{Q}(\tau_1))\, \eta_{\Delta_2}(\mathscr{Q}(\tau_2))\, \eta_{\Delta_2}(\mathscr{Q}(\tau_2))\, \eta_{\Delta_1}(\mathscr{Q}(\tau_1))\psi)$$

$$\to (\psi, T_{\tau_1}\{\eta_{\Delta_1}^2(T_{\tau_2-\tau_1}\eta_{\Delta_1}^2)\}(0,\mathscr{P})\psi) \tag{70}$$

Since (70) holds for all $\psi\in\mathscr{H}$ it follows that, *for all regions $\Delta_j$ from a countable set $\mathscr{R}$ and times $\tau_j$ from a countable set $\mathscr{T}$, with probability* 1,

$$\eta_{\Delta_{j_1}}(\mathscr{Q}(\tau_{j_1}))\, \eta_{\Delta_{j_2}}(\mathscr{Q}(\tau_{j_2}))\, \eta_{\Delta_{j_2}}(\mathscr{Q}(\tau_{j_2}))\, \eta_{\Delta_{j_1}}(\mathscr{Q}(\tau_{j_1}))$$

*converges weakly to* $T_{\tau_{j_1}}\{\eta^2_{\Delta_{j_1}}(T_{\tau_{j_2}-\tau_{j_1}}\eta^2_{\Delta_{j_1}})\}(0,\mathscr{P})$. *This extends to any miltiple-time position measurements with regions from* $\mathscr{R}$ *and times from* $\mathscr{T}$.

By shifting the particle by the macroscopic amount $X_0$, a general initial distribution $h(X_0)\,d(X_0)$ can be constructed (Section 2.7).[19] A similar formula holds for general multiple-time position measurements.

**The Boltzmann Equation.** If the initial probability distribution is $d\mu(x,p)=g(x,p)\,dx\,dp$, then Eq. (63) takes the form

$$\langle f(x(\tau),p(\tau))\rangle=\int dx\,dp[T_\tau^* g](x,p)\,f(x,p)$$

Hence the probability density at time $\tau$ is

$$g_\tau=T_\tau^*\,g \tag{71}$$

where $T_\tau^*$ is the Hilbert space adjoint in $L^2(R^\nu\times R^\nu, dx\,dp)$ of $T_\tau$. A straightforward calculation yields $T_\tau^{0*}=T^0_{-\tau}$ or $\mathscr{L}_0^*=-\mathscr{L}_0$. On the other hand, $\mathscr{L}_1^*=\mathscr{L}_1$, since $\mathscr{L}_1$ is a positive operator. [See the discussion of Eq. (74) below] Then $g_\tau$ satisfies the equation

$$\partial_\tau g_\tau(x,p)=-(1/m)\,p\cdot\partial_x g(x,p)+4\pi m/\hbar^{\nu+1}$$
$$\times\int dq\,\gamma((p-q)/\hbar)[g(x,q)-g(x,p)]\,\delta(q^2-p^2) \tag{72}$$

[See Eq. (73) below.]

Equation (72) has the form of the linear Boltzmann equation for the classical Lorentz gas given in ref. 20, eq. (2.28).

**The Limit** $\hbar\to 0$. The limit of the generator $\mathscr{L}_1$, (66), as $\hbar\to 0$ can be computed (ref. 6, § 7). We give a brief discussion, and drop the variable $x$, as it plays no role. First, with $q=p-\hbar k$,

$$(\mathscr{L}_1 f)(p)=4\pi m/\hbar^{\nu+1}\int dq\,\gamma((p-q)/\hbar)[f(p)-f(q)]\,\delta(q^2-p^2) \tag{73}$$

Then, using $\gamma(-k)=\gamma(k)$,[20] we have

$$\int dp\,g(p)(\mathscr{L}_1 f)(p)=2\pi m/\hbar^{\nu+1}\int dp\int dq\,\gamma((p-g)/\hbar)$$
$$\times[g(p)-g(q)][f(p)-f(q)]\,\delta(q^2-p^2) \tag{74}$$

[19] The translation invariance of the random field is used here.
[20] $\langle v(0)\,v(x)\rangle=\langle v(x)\,v(0)\rangle=\langle v(0)\,v(-x)\rangle$.

which, since $\gamma(k) \geqslant 0$, shows the positivity of the operator $\mathscr{L}_1$. Going back to the variable $k$,

$$\int dp\, g(p)(\mathscr{L}_1 f)(p)$$

$$= \pi m \int dk\, \gamma(k) \int dp[g(p) - g(p - \hbar k)]/\hbar[f(p) - f(p - \hbar k)]/$$
$$\hbar\delta(p \cdot k - \hbar k^2/2)$$

$$\to \pi m \int dk\, \gamma(k) \int dp\, k \cdot \nabla g(p) k \cdot \nabla f(p)\, \delta(p \cdot k)$$

$$= \int dp \sum_{\alpha\beta} A_{\alpha\beta}\, \partial_\alpha\, g(p)\, \partial_\beta f(p)$$

where

$$A_{\alpha\beta}(p) = \int dk\, \gamma(k)\, k_\alpha k_\beta\, \delta(p \cdot k)$$

Hence $\lim_{\hbar \to 0} \mathscr{L}_1 = \mathscr{L}'$, where

$$\mathscr{L}' f = -\sum_{\alpha\beta} \partial_\alpha (A_{\alpha\beta}\, \partial_\beta f)$$

Hence in the limit $\hbar \to 0$ the linear Boltzmann equation (72) goes over to the linear Landau equation [ref. 20, Eq. (2.19)]:

$$\partial_\tau\, g_\tau(x, p) = -(1/m) p \cdot \partial_x\, g(x, p) + \sum_{\alpha\beta} \partial_{p_\alpha} [A_{\alpha\beta}(p)\, \partial_{p_\beta} g(x, p)]$$

If the random potential is isotropic, so $\gamma(k) = \Gamma(|k|)$, then

$$A_{\alpha\beta}(p) = (\delta_{\alpha\beta} - p_\alpha p_\beta\, |p|^{-2})\, c/|p|$$

where $c = \pi \int d\zeta\, \zeta^3 \Gamma(\zeta)$ (for $\nu = 3$). These coefficients $A_{\alpha\beta}(p)$ have the form given in ref. 20, Eq. (2.20).

### 3.7. Many Particles

The weak convergence with probability 1 obtained in Section 3.6 immediately extends to any finite number $N$ of particles, using the fact that if, for $j = 1, \ldots, N$, $A_j(\lambda)$ converges weakly to $A_j$ in $\mathscr{H}$, then

$A_1(\lambda) \otimes \cdots \otimes A_N(\lambda)$ converges weakly to $A_1 \otimes \cdots \otimes A_n$ in $\mathscr{H} \otimes \cdots \otimes \mathscr{H}$. Thus the joint distribution of multiple-time position measurements on $N$ particles is given, in the large-space-time, weak-coupling limit, by the joint multiple-time position distribution of $N$ classical particles performing independent equivalent jump processes, with initial distribution

$$h(X_0^{(1)},..., X_0^{(N)}) \, |\hat{\psi}(p^{(1)},..., p^{(N)})|^2 \, dX_0^{(1)} \cdots dX_0^{(N)} \, dp^{(1)} \cdots dp^{(N)}$$

(We have included a macroscopic displacement of the particles as in Section 2.7. Otherwise, all particles start at $x=0$.)

If the quantum particles are initially uncorrelated, the wavefunction $\psi = \psi^{(1)} \otimes \cdots \otimes \psi^{(N)}$, and $h$ factorizes. It follows that in the large-space-time, weak-coupling limit, the classical particle trajectories will be uncorrelated.

The discussion of Bose and Fermi statistics leads to the same conclusion as in Section 2.7, but the argument is different since we have weak convergence and not strong convergence in the limit $\lambda \to 0$. We need to consider (52), where in the formula (43) for $\mathscr{F}(t_1,..., t_N)$ we include a factor $\exp[i\lambda^{-2} X_0 \cdot p]$. The upper bound (46) is unchanged, and hence dominated convergence again implies (53), where, however, now $\mathscr{F}_0(w, v) = 0$. This follows from the fact that, in the formula (43) for $\mathscr{F}(t_1,..., t_N)$,

$$\int dp \, \overline{\hat{\psi}(p)} \exp[i\lambda^{-2} X_0 \cdot p] \exp[-ip \cdot x/h]$$

$$\times \exp\left[ i(p/m) \cdot \left\{ \sum_{a=1}^{4} \tau_a \kappa_a + \sum_{l=1}^{N-M} [t_{\beta(l)} - t_{\alpha(l)}] \, k_{\beta(l)} \right\} \right] \to 0$$

as $\lambda \to 0$ by the Riemann-Lebesgue lemma. (A dominated convergence argument completes the proof.)

## 4. MACROSCOPIC LOCALIZATION OF RELATIVISTIC PARTICLES

The previous sections have dealt with the large-space-time description of non-relativistic quantum particles. The same procedure applies to free relativistic particles with Hamiltonian $H = (\mathscr{P}^2 c^2 + m^2 c^4)^{1/2}$ and indeed to a general dependence of energy $\varepsilon(p)$ on momentum $p$. Some difficulties in defining localization for relativistic particles (25, 18) are removed by considering the large-space-time limit. (See also ref. 22.) On a macroscopic scale, particle trajectories will be obtained which have the form $X(\tau) = X_0 + \tau v(p)$, where $v(p)$ is the group velocity

$$v(p) = \nabla_p \varepsilon(p)$$

In the relativistic case $v(p) = p/(m^2 + p^2/c^2)^{1/2}$. Since $|v(p)| \leqslant c$, the velocity operator $v(\mathscr{P})$ is bounded.

The wavefunctions of the particle will be given as square-integrable functions of momentum and the Hilbert space will be represented as $\mathscr{H} = L^2(P^3, d^3p)$. (Spin degrees of freedom are ignored, as they play no role in the present analysis.) The momentum operator $\mathscr{P}$ is multiplication by $p$ and the position operator is $q = i\hbar\partial_p$. This position operator does not transform in a Lorentz-covariant manner, but in the macroscopic limit Lorentz-covariant trajectories are obtained. An interpretation of this is that an apparatus to measure "microscopic position" has a complicated description with respect to a moving observer, but on a macroscopic scale a covariant transformation law is obtained.

The usual commutation relations hold between $q$ and $\mathscr{P}$, $[q_\alpha, \mathscr{P}_\beta] = i\hbar\delta_{\alpha\beta}$, and

$$\exp[ia \cdot q] \exp[ib \cdot \mathscr{P}] \exp[-ia \cdot q] = \exp[-i\hbar a \cdot b] \exp[ib \cdot \mathscr{P}]$$

Thus

$$\exp[ia \cdot q] \exp[-i(t/\hbar)\, \varepsilon(\mathscr{P})] \exp[-ia \cdot q] = \exp[-i(t/\hbar)\varepsilon(\mathscr{P} - \hbar a)]$$

Consequently,

$$\begin{aligned}\exp[ia \cdot q(t)] &= \exp[i(t/\hbar)\, \varepsilon(\mathscr{P})] \exp[ia \cdot q] \exp[-i(t/\hbar)\, \varepsilon(\mathscr{P})] \\ &= \exp[i(t/\hbar)\{\varepsilon(\mathscr{P}) - \varepsilon(\mathscr{P} - \hbar a)\}] \exp[ia \cdot q] \\ &= \exp\left[i(t/\hbar)a \cdot \int_0^\hbar ds\, v(\mathscr{P} - sa)\right] \exp[ia \cdot q]\end{aligned}$$

It follows on taking the limit $a \to 0$ that on the domain of $q$,

$$q(t) = q + tv(\mathscr{P})$$

The development now parallels the preceding discussion of nonrelativistic particles. The wavefunction of the particle may be given a macroscopic space-time displacement $\lambda^{-2}(X_0, \tau_0)$ using the unitary operator $\exp[(i/\hbar)\,\lambda^{-2}(\tau_0 H - X_0 \cdot \mathscr{P})]$ and this displacement may be shifted to the position observable:

$$q(t) \to \lambda^{-2}X_0 + q + (t - \lambda^{-2}\tau_0)\, v(\mathscr{P})$$

Now introduce the macroscopic time $\tau = \lambda^2 t$ and the macroscopic position operator $\mathscr{Q} = \lambda^2 q$. Then

$$\mathscr{Q}(\tau) = \lambda^2 q(\lambda^{-2}\tau) = \lambda^2 q + X_0 + (\tau - \tau_0)\, v(\mathscr{P})$$

In the limit $\lambda \to 0$, $\mathcal{Q}(\tau)$ converges strongly on the domain of $q$ to the bounded operator $X(\tau)$:

$$\mathcal{Q}(\tau) \to X(\tau) = X_0 + (\tau - \tau_0) v(\mathcal{P}) \qquad \text{as} \quad \lambda \to 0$$

and hence for any bounded function $F$ which is continuous except on a set of Lebesgue measure 0,

$$F(\mathcal{Q}(\tau)) \to F(X_0 + (\tau - \tau_0) v(\mathcal{P})) \qquad \text{as} \quad \lambda \to 0$$

The joint distribution for multiple-time position measurements will then be given by the trajectories $X(\tau) = X_0 + (\tau - \tau_0)\, v(p)$ with probability density $|\psi(p)|^2$.

**Remark.** The Lorentz covariance of the macroscopic position operator may be shown as follows. The wavefunction with the space-time displacement $\lambda^{-2}(\tau_0, X_0)$ is

$$\psi_{(\tau_0, X_0)} = \exp[(i/\hbar)\, \lambda^{-2}(\tau_0, X_0) \cdot (H, c^2\mathcal{P})]\, \psi$$

where the Minkowski scalar product is

$$(A_0, \underline{A}) \cdot (B_0, \underline{B}) = A_0 B_0 - c^{-2} \underline{A} \cdot \underline{B}$$

and $(H, c^2\mathcal{P})$ transforms as a four-vector:

$$U(\Lambda)^{-1}\, (H, c^2\mathcal{P})\, U(\Lambda) = \Lambda(H, c^2\mathcal{P})$$

The Lorentz transformation $\Lambda$ applied to $\psi_{(\tau_0, X_0)}$ is

$$\begin{aligned} U(\Lambda)\, \psi_{(\tau_0, X_0)} &= \exp[(i/\hbar)\, \lambda^{-2}(\tau_0, X_0) \cdot \Lambda^{-1}(H, c^2\mathcal{P})]\; U(\Lambda)\psi \\ &= \exp[(i/\hbar)\, \lambda^{-2}(\tau_0', X_0') \cdot (H, c^2\mathcal{P})]\; U(\Lambda)\psi \end{aligned}$$

where $(\tau_0', X_0') = \Lambda(\tau_0, X_0)$. Now transferring the displacement $\lambda^{-2}(\tau_0', X_0')$ to the observables gives

$$\mathcal{Q}(\tau) = \lambda^2 q + (\tau - \tau_0')\, v(\mathcal{P}) + X_0'$$

and now transferring the Lorentz transformation to the observables gives

$$\mathcal{Q}(\tau) = \lambda^2 U(\Lambda)^{-1} q U(\Lambda) + (\tau - \tau_0')\, v(\mathcal{P}') + X_0'$$

where $\mathcal{P}'$ is the Lorentz-transformed momentum. In the limit $\lambda \to 0$, $\mathcal{Q}(\tau)$ converges to

$$X_0' + (\tau - \tau_0') v(\mathcal{P}')$$

which just corresponds to the Lorentz-transformed trajectories.

## REFERENCES

1. J. S. Bell, On wave packet reduction in the Coleman–Hepp model, *Helv. Phys. Acta* **48**:93 (1975).
2. R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (Wiley, 1975).
3. R. Carmona and J. Lacroix, *Special Theory of Random Schrodinger Operators* (Birkhaüser, 1990).
4. C. G. Darwin, Free motion in the wave mechanics, *Proc. Soc.* **117A**:258 (1927).
5. E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, 1976).
6. G. F. Dell' Antonio, large time small coupling behaviour of a quantum particle in a random potential, *Ann. Inst. Henri Poincaré A* **39**:339 (1983).
7. D. Dürr, S. Goldstein, and J. L. Lebowitz, Asymptotic motion of a classical particle in a random potential in two dimensions: Landau model, *Commun. Math. Phys.* **113**:209 (1987).
8. W. G. Faris, *Self-Adjoint Operators* (Springer, 1975).
9. W. Heisenberg, *The Physical Principles of the Quantum Theory* (Dover, New York).
10. K. Hepp, Quantum theory of measurement and macroscopic observables, *Helv. Phys. Acta* **45**:237 (1972).
11. K. Hepp, The classical limit for quantum mechanical correlation functions, *Commun. Math. Phys.* **35**:265 (1974).
12. T. G. Ho, L. J. Landau, and A. J. Wilkins, On the weak coupling limit for a Fermi gas in a random potential, *Rev. Math. Phys.* **5**(2):209 (1993).
13. K. Kraus, *States, Effects, and Operations* (Springer, 1983).
14. H. Kesten and G. C. Papanicolau, A limit theorem for stochastic acceleration, *Commun. Math. Phys.* **78**:19 (1980).
15. L. J. Landau, Weak coupling limit: Feynman diagrams, in *NATO ASI Series Vol. 324 "On Three Levels,"* eds. M. Fannes, C. Maes, A. Verbeure (Plenum Press, New York, 1994).
16. N. F. Mott, The wave mechanics of α-ray tracks, *Proc. R. Soc. A* **126**:79 (1929).
17. P. Martin and G. G. Emch, A rigorous model sustaining van Hove's phenomenon, *Helv. Phys. Acta* **48**:59 (1975).
18. J. F. Perez and I. F. Wilde, Localization and causality in relativistic quantum mechanics, *Phys. Rev. D* **16**:315 (1977).
19. H. Spohn, Derivation of the transport equation for electrons moving through random impurities, *J. Stat. Phys.* **17**:385 (1977).
20. H. Spohn, Kinetic equations from Hamiltonian dynamics: Markovian limits, *Rev. Mod. Phys.* **53**:569 (1980).
21. M. D. Srinivas and E. B. Davies, Photon counting Probabilities in quantum optics, *Optica Acta* **28**:981 (1981).
22. O. Steinmann, Particle localization in field theory, *Commun. Math. Phys.* **7**:112 (1968).
23. D. W. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes* (Springer, 1979).
24. E. H. Wichmann, Scattering of wave packets, *Am. J. Phys.* **33**:20 (1965).
25. A. S. Wightman, On the localizability of quantum mechanical system, *Rev. Mod. Phys.* **34**:845 (1962).
26. B. Whitten-Wolfe and G. G. Emch, A mechanical quantum measuring process, *Helv. Phys. Acta* **49**:45 (1976).